Complex Dynamics of Birational Surface Maps Defined over Number Fields

Paul Reschke

University of Michigan

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An Example (Favre; Bedford; Diller, Dujardin, Guedj)

For $\theta \in [0,1]$, we define a birational self-map on $\mathbb{P}^2$ by

$$f : [x : y : z] \mapsto [y^2 : 2\cos(\pi\theta)y^2 + z^2 - xy : yz]$$

$$\big(\text{so } f^{-1} : [x : y : z] \mapsto [2\cos(\pi\theta)x^2 + z^2 - xy : x^2 : xz]\big).$$

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We can conjugate $f|_{z=0}$ to

$$\phi \circ f|_{z=0} \circ \phi^{-1} : [x : y] \mapsto [e^{i2\pi\theta}x : y],$$

$$\text{with } \phi([1:0]) = [1:1] \;\&\; \phi([0:1]) = [e^{i2\pi\theta} : 1].$$

Example Continued - Stability/Energy Conditions

The condition for *algebraic stability*

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$$\sum_{n\geq 0} \lambda(f)^{-n} \log \operatorname{dist}(f^n(I(f^{-1})), I(f)) > -\infty \qquad \text{(BD)}$$

guarantees that f has a natural measure of maximal entropy with nice dynamical properties. (Bedford, Smillie, Lyubich, Cantat, . . .)

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Buff: $\exists\, \theta \notin \mathbb{Q}$ for which the nice dynamical properties fail to hold.

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So the sum in (BD) is no worse than

$$\sum_{n\geq 0} 2^{-n}(\log(C)-n\epsilon),$$

which converges.

A General Result

Theorem (Jonsson, R.)

Let f be a birational self-map on a smooth complex projective surface X, and suppose $\lambda(f) > 1$. If X and f are defined over a number field, then there is a smooth birational model of X on which f satisfies (BD).

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Diller, Favre:
assuming (AS), $\exists$ nef $L^+ \in \mathrm{Pic}(X)_{\mathbb{R}}$ such that $f^*L^+ = \lambda(f)L^+$; if f is not conjugate to an automorphism, then $(L^+ \cdot L^+) > 0$ (i.e., L^+ is big).

Argument in the Case $X = \mathbb{P}^2$, part 1

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$$h_{x,v} : [x : y : z] \mapsto \log \max\{1, |y/x|_v, |z/x|_v\} \ \& \ h_{y,v} \ \& \ h_{z,v}.$$

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(which is independent of the choice of $\alpha \in \{x, y, z\}$).

Let $\infty \in M_K$ denote the archimedean place from the implicitly given embedding $K \hookrightarrow \mathbb{C}$, and set

$$\psi_{\alpha,\infty}([x : y : z]) := h_{\alpha,\infty}(f([x : y : z])) - h_{f^*\alpha,\infty}([x : y : z]).$$

Argument in the Case $X = \mathbb{P}^2$, part 2

Writing $f([x : y : z]) = [f_x, f_y, f_z]$, where f_x, f_y, and f_z are homogeneous polynomials of degree $\lambda(f)$ in x, y, and z, we have

$$\psi_{\alpha,\infty} : [x : y : z] \mapsto \log \frac{\max\{|f_x|_\infty, |f_y|_\infty, |f_z|_\infty\}}{\max\{|x^2|_\infty, |y^2|_\infty, |z^2|_\infty\}},$$

which (independent of α) is well-defined and bounded above on $\mathbb{P}^2 \setminus I(f)$.

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Also, $\exists\, D$ such that

$$\psi_{\alpha,\infty}([x:y:z]) \leq \log \operatorname{dist}([x:y:z], I(f)) + D$$

for all α and $[x:y:z] \in \mathbb{P}^2 \setminus \{f^*\alpha = 0\}$. (Note that $I(f) \subseteq \{f^*\alpha = 0\}$.)

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For each $v \in M_K$, $\exists\, C_v \geq 0$ such that $\psi_{\alpha,v}$ is bounded above by C_v on $\mathbb{P}^2 \setminus I(f)$; moreover, we can take $C_v = 0$ for all but finitely many v.

Argument in the Case $X = \mathbb{P}^2$, part 3

For $p = [x : y : z] \in I(f^{-1})$, consider

$$\lambda(f)^{-n} h_{L^+}(f^n(p)) - h_{L^+}(p) =$$

$$\sum_{k=0}^{n-1} \sum_{v \in M_K} \psi_v(f^k(p)) + \sum_{k=0}^{n-1} h_{f^* L^+}(f^k(p)) - \lambda(f) h_{L^+}(f^k(p)).$$

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The argument is essentially the same in all cases where L^+ is ample.

We get as a corollary that

$$\lim_{n \to \infty} \lambda(f)^{-n} h_{L^+}(f^n(p))$$

exists and is non-negative for every point whose forward orbit misses $I(f)$.

Key Steps in the Case Where L^+ Is Big but Not Ample

Kawaguchi: $L^+ = A + \sum \delta_j [C_j]$, with A Kähler, each $\delta_j > 0$, and each C_j a prime divisor satisfying $(L^+ \cdot [C_j]) = 0$.

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We show that each $[C_j]$ is either periodic for f^* or has the property that $(f^n)^*[C_j]$ is nef for $n >> 0$. So

$$L^+ = \sum \gamma_j G_j + N + P,$$

with N nef, P periodic for f^*, each $\gamma_j > 0$, and each G_j globally generated.

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Then we again consider

$$\lambda(f)^{-n} h_{L^+}(f^n(p)) - h_{L^+}(p).$$

thank you