The average number of integral points in orbits

Wade Hindes

CUNY Graduate Center

Joint Mathematics Meeting 2016
Seattle, WA.
February 8th, 2016

Integral Points in Orbits:

- Let K be a global field,

Integral Points in Orbits:

- Let K be a global field,
- $\mathcal{O}_K$ be the ring of integers of K,

Integral Points in Orbits:

- Let K be a global field,
- $\mathcal{O}_K$ be the ring of integers of K,
- $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree $d \geq 2$,

Integral Points in Orbits:

- Let K be a global field,
- $\mathcal{O}_K$ be the ring of integers of K,
- $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree $d \geq 2$,
- $\phi^n := \underbrace{\phi \circ \phi \circ \cdots \circ \phi}_{n}$ be the n^{th} iterate of ϕ,

Integral Points in Orbits:

- Let K be a global field,
- $\mathcal{O}_K$ be the ring of integers of K,
- $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree $d \geq 2$,
- $\phi^n := \underbrace{\phi \circ \phi \circ \cdots \circ \phi}_{n}$ be the n^{th} iterate of ϕ,
- $\mathcal{O}_\phi^+(P) := \{P, \phi(P), \phi^2(P) \dots\}$ be the *orbit* of $P \in \mathbb{P}^1(\overline{K})$.

Integral Points in Orbits:

- Let K be a global field,
- $\mathcal{O}_K$ be the ring of integers of K,
- $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree $d \geq 2$,
- $\phi^n := \underbrace{\phi \circ \phi \circ \cdots \circ \phi}_{n}$ be the n^{th} iterate of ϕ,
- $\mathcal{O}_\phi^+(P) := \{P, \phi(P), \phi^2(P) \dots\}$ be the *orbit* of $P \in \mathbb{P}^1(\overline{K})$.

Goal: Given $P \in \mathbb{P}^1(K)$, we want to understand $\mathcal{O}_\phi^+(P) \cap \mathcal{O}_K$.

Integral Points in Orbits:

- Let K be a global field,
- $\mathcal{O}_K$ be the ring of integers of K,
- $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree $d \geq 2$,
- $\phi^n := \underbrace{\phi \circ \phi \circ \cdots \circ \phi}_{n}$ be the n^{th} iterate of ϕ,
- $\mathcal{O}_\phi^+(P) := \{P, \phi(P), \phi^2(P) \ldots\}$ be the *orbit* of $P \in \mathbb{P}^1(\overline{K})$.

Goal: Given $P \in \mathbb{P}^1(K)$, we want to understand $\mathcal{O}_\phi^+(P) \cap \mathcal{O}_K$.

Basic questions:

1. For which ϕ is $\mathcal{O}_\phi^+(P) \cap \mathcal{O}_K$ finite for all $P \in \mathbb{P}^1(K)$?

Integral Points in Orbits:

- Let K be a global field,
- $\mathcal{O}_K$ be the ring of integers of K,
- $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree $d \geq 2$,
- $\phi^n := \underbrace{\phi \circ \phi \circ \cdots \circ \phi}_{n}$ be the n^{th} iterate of ϕ,
- $\mathcal{O}_\phi^+(P) := \{P, \phi(P), \phi^2(P) \dots\}$ be the *orbit* of $P \in \mathbb{P}^1(\overline{K})$.

Goal: Given $P \in \mathbb{P}^1(K)$, we want to understand $\mathcal{O}_\phi^+(P) \cap \mathcal{O}_K$.

Basic questions:

1. For which ϕ is $\mathcal{O}_\phi^+(P) \cap \mathcal{O}_K$ finite for all $P \in \mathbb{P}^1(K)$?
2. When finite, how large is $\mathcal{O}_\phi^+(P) \cap \mathcal{O}_K$?

Integral Points in Orbits:

- Let K be a global field,
- $\mathcal{O}_K$ be the ring of integers of K,
- $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree $d \geq 2$,
- $\phi^n := \underbrace{\phi \circ \phi \circ \cdots \circ \phi}_{n}$ be the n^{th} iterate of ϕ,
- $\mathcal{O}_\phi^+(P) := \{P, \phi(P), \phi^2(P) \dots\}$ be the <u>*orbit*</u> of $P \in \mathbb{P}^1(\overline{K})$.

Goal: Given $P \in \mathbb{P}^1(K)$, we want to understand $\mathcal{O}_\phi^+(P) \cap \mathcal{O}_K$.

Basic questions:

1. For which ϕ is $\mathcal{O}_\phi^+(P) \cap \mathcal{O}_K$ finite for all $P \in \mathbb{P}^1(K)$?
2. When finite, how large is $\mathcal{O}_\phi^+(P) \cap \mathcal{O}_K$? Explicit bounds?

Example: Let $K = \mathbb{Q}$ and $\mathcal{O}_K = \mathbb{Z}$. Then we set

$$\phi(x) = \frac{899x^2 - 2002x + 275}{33x^2 - 584x + 275}$$

Example: Let $K = \mathbb{Q}$ and $\mathcal{O}_K = \mathbb{Z}$. Then we set

$$\phi(x) = \frac{899x^2 - 2002x + 275}{33x^2 - 584x + 275}$$

and check that

$$\mathcal{O}_\phi^+(0) = \left\{0, 1, 3, -2, 5, -7, \frac{2917}{299}, \frac{-296398306}{10198813} \dots \right\}.$$

Example: Let $K = \mathbb{Q}$ and $\mathcal{O}_K = \mathbb{Z}$. Then we set

$$\phi(x) = \frac{899x^2 - 2002x + 275}{33x^2 - 584x + 275}$$

and check that

$$\mathcal{O}_\phi^+(0) = \left\{0, 1, 3, -2, 5, -7, \frac{2917}{299}, \frac{-296398306}{10198813} \dots \right\}.$$

When finiteness fails:

1. **Polynomials:** If $\phi \in \mathcal{O}_K[x]$ and $P \in \mathcal{O}_K$ has infinte orbit, then $\mathcal{O}_\phi^+(P) \cap \mathcal{O}_K$ is infinite.

Example: Let $K = \mathbb{Q}$ and $\mathcal{O}_K = \mathbb{Z}$. Then we set

$$\phi(x) = \frac{899x^2 - 2002x + 275}{33x^2 - 584x + 275}$$

and check that

$$\mathcal{O}_\phi^+(0) = \left\{0, 1, 3, -2, 5, -7, \frac{2917}{299}, \frac{-296398306}{10198813} \dots \right\}.$$

<u>**When finiteness fails:**</u>

1. **Polynomials:** If $\phi \in \mathcal{O}_K[x]$ and $P \in \mathcal{O}_K$ has infinte orbit, then $\mathcal{O}_\phi^+(P) \cap \mathcal{O}_K$ is infinite.

2. **Polynomial Iterates:** Let $\phi(x) = 1/x^2$ and $P = 2$. Then

$$\mathcal{O}_\phi^+(P) \cap \mathbb{Z} = \{2, 2^4, 2^{16}, \dots\}.$$

Example: Let $K = \mathbb{Q}$ and $\mathcal{O}_K = \mathbb{Z}$. Then we set

$$\phi(x) = \frac{899x^2 - 2002x + 275}{33x^2 - 584x + 275}$$

and check that

$$\mathcal{O}_\phi^+(0) = \left\{0, 1, 3, -2, 5, -7, \frac{2917}{299}, \frac{-296398306}{10198813} \dots \right\}.$$

When finiteness fails:

1. **Polynomials:** If $\phi \in \mathcal{O}_K[x]$ and $P \in \mathcal{O}_K$ has infinte orbit, then $\mathcal{O}_\phi^+(P) \cap \mathcal{O}_K$ is infinite.

2. **Polynomial Iterates:** Let $\phi(x) = 1/x^2$ and $P = 2$. Then

$$\mathcal{O}_\phi^+(P) \cap \mathbb{Z} = \{2, 2^4, 2^{16}, \dots\}.$$

In this case, $\phi^2 = x^4 \in \mathbb{Z}[x]$.

Question: How do you check if $\phi^n \in \overline{K}[x]$ for some $n \geq 1$?

Question: How do you check if $\phi^n \in \overline{K}[x]$ for some $n \geq 1$?

Fact: If $\phi^n \in \overline{K}[x]$, then $\phi^2 \in \overline{K}[x]$.

Question: How do you check if $\phi^n \in \overline{K}[x]$ for some $n \geq 1$?

Fact: If $\phi^n \in \overline{K}[x]$, then $\phi^2 \in \overline{K}[x]$. (Riemann-Hurwitz formula)

Question: How do you check if $\phi^n \in \overline{K}[x]$ for some $n \geq 1$?

Fact: If $\phi^n \in \overline{K}[x]$, then $\phi^2 \in \overline{K}[x]$. (Riemann-Hurwitz formula)

Theorem (Silverman): Let K be a number field. If $\phi^2 \notin \overline{K}[x]$, then:

1. $\mathcal{O}_\phi^+(P) \cap \mathcal{O}_K$ is finite for all $P \in \mathbb{P}^1(K)$.

Question: How do you check if $\phi^n \in \overline{K}[x]$ for some $n \geq 1$?

Fact: If $\phi^n \in \overline{K}[x]$, then $\phi^2 \in \overline{K}[x]$. (Riemann-Hurwitz formula)

Theorem (Silverman): Let K be a number field. If $\phi^2 \notin \overline{K}[x]$, then:

1. $\mathcal{O}_\phi^+(P) \cap \mathcal{O}_K$ is finite for all $P \in \mathbb{P}^1(K)$.
2. Let $K = \mathbb{Q}$ and $\phi^n(P) = a_n/b_n$.

Question: How do you check if $\phi^n \in \overline{K}[x]$ for some $n \geq 1$?

Fact: If $\phi^n \in \overline{K}[x]$, then $\phi^2 \in \overline{K}[x]$. (Riemann-Hurwitz formula)

Theorem (Silverman): Let K be a number field. If $\phi^2 \notin \overline{K}[x]$, then:

1. $\mathcal{O}_\phi^+(P) \cap \mathcal{O}_K$ is finite for all $P \in \mathbb{P}^1(K)$.
2. Let $K = \mathbb{Q}$ and $\phi^n(P) = a_n/b_n$. If $\mathcal{O}_\phi^+(P)$ is infinite, then

$$\lim_{n\to\infty} \frac{\log|a_n|}{\log|b_n|} = 1$$

Back to our (naïve) questions:

Back to our (naïve) questions:

1. How large can $\#(\mathcal{O}_\phi^+(P) \cap \mathcal{O}_K)$ be?

Back to our (naïve) questions:

1. How large can $\#(\mathcal{O}_\phi^+(P) \cap \mathcal{O}_K)$ be?
2. Are there bounds that depend only on d and K?

Back to our (naïve) questions:

1. How large can $\#(\mathcal{O}_\phi^+(P) \cap \mathcal{O}_K)$ be?
2. Are there bounds that depend only on d and K?

Proposition: For all $N \geq 0$ and $d \geq 2$, there exists $\phi \in \mathbb{Q}(x)$ of degree d satisfying:

1. $\phi^2 \notin \overline{\mathbb{Q}}(x)$,
2. $\mathcal{O}_\phi^+(0)$ is infinite,
3. $\#(\mathcal{O}_\phi^+(0) \cap \mathbb{Z}) \geq N$.

Back to our (naïve) questions:

1. How large can $\#(\mathcal{O}_\phi^+(P) \cap \mathcal{O}_K)$ be?
2. Are there bounds that depend only on d and K?

Proposition: For all $N \geq 0$ and $d \geq 2$, there exists $\phi \in \mathbb{Q}(x)$ of degree d satisfying:

1. $\phi^2 \notin \overline{\mathbb{Q}}(x)$,
2. $\mathcal{O}_\phi^+(0)$ is infinite,
3. $\#(\mathcal{O}_\phi^+(0) \cap \mathbb{Z}) \geq N$.

Refined conjecture: There is a bound on $\#(\mathcal{O}_\phi^+(P) \cap \mathcal{O}_K)$ that depends only on d, K, and $\mathfrak{R}\phi$ (the minimal resultant).

Back to our (naïve) questions:

1. How large can $\#(\mathcal{O}_\phi^+(P) \cap \mathcal{O}_K)$ be?
2. Are there bounds that depend only on d and K?

Proposition: For all $N \geq 0$ and $d \geq 2$, there exists $\phi \in \mathbb{Q}(x)$ of degree d satisfying:

1. $\phi^2 \notin \overline{\mathbb{Q}}(x)$,
2. $\mathcal{O}_\phi^+(0)$ is infinite,
3. $\#(\mathcal{O}_\phi^+(0) \cap \mathbb{Z}) \geq N$.

Refined conjecture: There is a bound on $\#(\mathcal{O}_\phi^+(P) \cap \mathcal{O}_K)$ that depends only on d, K, and $\mathfrak{R}\phi$ (the minimal resultant).

This is analogous to a conjecture of Lang on integral points on elliptic curves.

Easier Question: How large to we expect $\#(\mathcal{O}_\phi^+(P) \cap \mathcal{O}_K)$ to be if we choose P at "*random*" ?

Easier Question: How large to we expect $\#(\mathcal{O}_\phi^+(P) \cap \mathcal{O}_K)$ to be if we choose P at "*random*" ?

A reasonable interpretation of this question can be stated in terms of averages.

Easier Question: How large to we expect $\#(\mathcal{O}_\phi^+(P) \cap \mathcal{O}_K)$ to be if we choose P at "*random*" ?

A reasonable interpretation of this question can be stated in terms of averages.

Definition: For all $B > 0$, let $\mathbb{P}^1(B, K)$ be the set of points in $\mathbb{P}^1(K)$ of height at most B (finite by Northcott)

Easier Question: How large to we expect $\#(\mathcal{O}_\phi^+(P) \cap \mathcal{O}_K)$ to be if we choose P at "*random*" ?

A reasonable interpretation of this question can be stated in terms of averages.

Definition: For all $B > 0$, let $\mathbb{P}^1(B, K)$ be the set of points in $\mathbb{P}^1(K)$ of height at most B (finite by Northcott) and define

$$\overline{\mathrm{Avg}}(\phi, S) := \limsup_{B \to \infty} \frac{\sum_{P \in \mathbb{P}^1(B,K)} \#\left(\mathcal{O}_\phi^+(P) \cap \mathcal{O}_{K,S}\right)}{\#\mathbb{P}^1(B, K)}.$$

for any finite set of places S of K.

Easier Question: How large to we expect $\#(\mathcal{O}_\phi^+(P) \cap \mathcal{O}_K)$ to be if we choose P at "*random*" ?

A reasonable interpretation of this question can be stated in terms of averages.

Definition: For all $B > 0$, let $\mathbb{P}^1(B, K)$ be the set of points in $\mathbb{P}^1(K)$ of height at most B (finite by Northcott) and define

$$\overline{\text{Avg}}(\phi, S) := \limsup_{B \to \infty} \frac{\sum_{P \in \mathbb{P}^1(B,K)} \#\big(\mathcal{O}_\phi^+(P) \cap \mathcal{O}_{K,S}\big)}{\#\mathbb{P}^1(B,K)}.$$

for any finite set of places S of K.

Goal: Bound $\overline{\text{Avg}}(\phi, S)$ independently of ϕ (even d and S)

Theorem (WH 2015)

If K is a number field, $\phi^2 \notin \overline{K}[x]$ and $\deg(\phi) \geq 2$, then

$$\overline{\text{Avg}}(\phi, S) := \limsup_{B\to\infty} \frac{\sum_{P\in\mathbb{P}^1(B,K)} \#\left(\mathcal{O}_\phi^+(P) \cap \mathcal{O}_{K,S}\right)}{\#\mathbb{P}^1(B,K)} = 0$$

for all S.

Theorem (WH 2015)

If K is a number field, $\phi^2 \notin \overline{K}[x]$ and $\deg(\phi) \geq 2$, then

$$\overline{\mathrm{Avg}}(\phi, S) := \limsup_{B\to\infty} \frac{\sum_{P\in\mathbb{P}^1(B,K)} \#\left(\mathcal{O}_\phi^+(P) \cap \mathcal{O}_{K,S}\right)}{\#\mathbb{P}^1(B,K)} = 0$$

for all S. If in addition, ϕ is separable and $\infty \in \mathrm{Per}(\phi)$, then $\overline{\mathrm{Avg}}(\phi, S) = 0$ when $K/\mathbb{F}_q(t)$.

Theorem (WH 2015)

If K is a number field, $\phi^2 \notin \overline{K}[x]$ and $\deg(\phi) \geq 2$, then

$$\overline{\text{Avg}}(\phi, S) := \limsup_{B \to \infty} \frac{\sum_{P \in \mathbb{P}^1(B,K)} \#\left(\mathcal{O}_\phi^+(P) \cap \mathcal{O}_{K,S}\right)}{\#\mathbb{P}^1(B,K)} = 0$$

for all S. If in addition, ϕ is separable and $\infty \in \text{Per}(\phi)$, then $\overline{\text{Avg}}(\phi, S) = 0$ when $K/\mathbb{F}_q(t)$.

Generalization: We can generalize this result to 1-dimensional families of orbits, assuming a height uniformity conjecture in arithmetic geometry.

Integral points in families of orbits:

- Let $X_{/K}$ be a variety,

Integral points in families of orbits:

- Let $X_{/K}$ be a variety,
- Let $h_X := h_{X,D} : X(\overline{K}) \to \mathbb{R}_{\geq 0}$ be an ample height function,

Integral points in families of orbits:

- Let $X_{/K}$ be a variety,
- Let $h_X := h_{X,D} : X(\overline{K}) \to \mathbb{R}_{\geq 0}$ be an ample height function,
- Let $\mathrm{Rat}_d \subseteq \mathbb{P}^{2d-1}$ be the space of rational maps of degree d,

Integral points in families of orbits:

- Let $X_{/K}$ be a variety,
- Let $h_X := h_{X,D} : X(\overline{K}) \to \mathbb{R}_{\geq 0}$ be an ample height function,
- Let $\mathrm{Rat}_d \subseteq \mathbb{P}^{2d-1}$ be the space of rational maps of degree d,
- Let $\phi : X \to \mathrm{Rat}_d$ and $\beta : X \to \mathbb{P}^1$ be rational maps (over K).

Integral points in families of orbits:

- Let $X_{/K}$ be a variety,
- Let $h_X := h_{X,D} : X(\overline{K}) \to \mathbb{R}_{\geq 0}$ be an ample height function,
- Let $\mathrm{Rat}_d \subseteq \mathbb{P}^{2d-1}$ be the space of rational maps of degree d,
- Let $\phi : X \to \mathrm{Rat}_d$ and $\beta : X \to \mathbb{P}^1$ be rational maps (over K).

For $P \in X(\overline{K})$, we let ϕ_P (resp. β_P) be the image of P under ϕ (and β resp.)

Integral points in families of orbits:

- Let $X_{/K}$ be a variety,
- Let $h_X := h_{X,D} : X(\overline{K}) \to \mathbb{R}_{\geq 0}$ be an ample height function,
- Let $\mathrm{Rat}_d \subseteq \mathbb{P}^{2d-1}$ be the space of rational maps of degree d,
- Let $\phi : X \to \mathrm{Rat}_d$ and $\beta : X \to \mathbb{P}^1$ be rational maps (over K).

For $P \in X(\overline{K})$, we let ϕ_P (resp. β_P) be the image of P under ϕ (and β resp.)

Goal: Study the distribution of $\#(\mathcal{O}^+_{\phi_P}(\beta_P) \cap \mathcal{O}_{K,S})$ as we vary P.

Integral points in families of orbits:

- Let $X_{/K}$ be a variety,
- Let $h_X := h_{X,D} : X(\overline{K}) \to \mathbb{R}_{\geq 0}$ be an ample height function,
- Let $\mathrm{Rat}_d \subseteq \mathbb{P}^{2d-1}$ be the space of rational maps of degree d,
- Let $\phi : X \to \mathrm{Rat}_d$ and $\beta : X \to \mathbb{P}^1$ be rational maps (over K).

For $P \in X(\overline{K})$, we let ϕ_P (resp. β_P) be the image of P under ϕ (and β resp.)

Goal: Study the distribution of $\#(\mathcal{O}_{\phi_P}^{+}(\beta_P) \cap \mathcal{O}_{K,S})$ as we vary P.

To do this, let $X(B, K)$ be the set of K-points of X of height at most B.

As before, we study the average:

$$\overline{\mathrm{Avg}}(\phi,\beta,S) := \limsup_{B\to\infty} \frac{\sum_{P\in X(B,K)} \#\big(\mathcal{O}_{\phi_P}^{+}(\beta_P)\cap\mathcal{O}_{K,S}\big)}{\#X(B,K)}.$$

As before, we study the average:

$$\overline{\mathrm{Avg}}(\phi, \beta, S) := \limsup_{B\to\infty} \frac{\sum_{P\in X(B,K)} \#\big(\mathcal{O}^+_{\phi_P}(\beta_P) \cap \mathcal{O}_{K,S}\big)}{\#X(B,K)}.$$

However, in order to prove an average-zero result, we need the following height uniformity conjecture.

As before, we study the average:

$$\overline{\text{Avg}}(\phi, \beta, S) := \limsup_{B \to \infty} \frac{\sum_{P \in X(B,K)} \#\left(\mathcal{O}^+_{\phi_P}(\beta_P) \cap \mathcal{O}_{K,S}\right)}{\#X(B,K)}.$$

However, in order to prove an average-zero result, we need the following height uniformity conjecture.

Conjecture: Let $\pi : \mathcal{C} \to \mathcal{B}$ be a family of curves of genus ≥ 2. Then there exist constants κ_1 and κ_2 such that

$$h_C(Q) \leq \kappa_1 \cdot h_{\mathcal{B}}(P) + \kappa_2$$

for all points $Q \in \mathcal{C}_P(K)$.

As before, we study the average:

$$\overline{\mathrm{Avg}}(\phi, \beta, S) := \limsup_{B \to \infty} \frac{\sum_{P \in X(B,K)} \#\left(\mathcal{O}^+_{\phi_P}(\beta_P) \cap \mathcal{O}_{K,S}\right)}{\#X(B,K)}.$$

However, in order to prove an average-zero result, we need the following height uniformity conjecture.

Conjecture: Let $\pi : \mathcal{C} \to \mathcal{B}$ be a family of curves of genus ≥ 2. Then there exist constants κ_1 and κ_2 such that

$$h_C(Q) \leq \kappa_1 \cdot h_{\mathcal{B}}(P) + \kappa_2$$

for all points $Q \in \mathcal{C}_P(K)$.

(Here $h_{\mathcal{C}}$ is an arbitrary height function on $\mathcal{C}$ and $h_{\mathcal{B}}$ is ample.)

As before, we study the average:

$$\overline{\text{Avg}}(\phi, \beta, S) := \limsup_{B\to\infty} \frac{\sum_{P\in X(B,K)} \#\big(\mathcal{O}^+_{\phi_P}(\beta_P) \cap \mathcal{O}_{K,S}\big)}{\#X(B,K)}.$$

However, in order to prove an average-zero result, we need the following height uniformity conjecture.

Conjecture: Let $\pi : \mathcal{C} \to \mathcal{B}$ be a family of curves of genus ≥ 2. Then there exist constants κ_1 and κ_2 such that

$$h_C(Q) \leq \kappa_1 \cdot h_{\mathcal{B}}(P) + \kappa_2$$

for all points $Q \in \mathcal{C}_P(K)$.

(Here $h_{\mathcal{C}}$ is an arbitrary height function on $\mathcal{C}$ and $h_{\mathcal{B}}$ is ample.)

This conjecture follows from the Vojta (main) conjecture.

Theorem (WH 2015)

Let $X_{/K}$ be a curve and suppose that $\phi : X \to \mathrm{Rat}_d$ and $\beta : X \to \mathbb{P}^1$ satisfy

$$\deg(\beta) > \frac{2d-1}{d-1} \cdot \deg(\phi^* H)$$

for the hyperplane class $H \in \mathrm{Pic}(\mathbb{P}^{2d+1})$. Then:

1. $\#\big(\mathcal{O}^+_{\phi_P}(\beta_P) \cap \mathcal{O}_{K,S}\big)$ *is bounded over all $P \in X(K)$.*
2. $\overline{\mathrm{Avg}}(\phi, \beta, S)$ *exists*
3. *Assuming the height uniformity conjecture, $\overline{\mathrm{Avg}}(\phi, \beta, S) = 0$.*

Theorem (WH 2015)

Let $X_{/K}$ be a curve and suppose that $\phi : X \to \text{Rat}_d$ and $\beta : X \to \mathbb{P}^1$ satisfy

$$\deg(\beta) > \frac{2d-1}{d-1} \cdot \deg(\phi^* H)$$

for the hyperplane class $H \in \text{Pic}(\mathbb{P}^{2d+1})$. Then:

1. $\#\big(\mathcal{O}^+_{\phi_P}(\beta_P) \cap \mathcal{O}_{K,S}\big)$ *is bounded over all $P \in X(K)$.*
2. $\overline{\text{Avg}}(\phi, \beta, S)$ *exists*
3. *Assuming the height uniformity conjecture,* $\overline{\text{Avg}}(\phi, \beta, S) = 0$.

Remarks:

1. This theorem holds unconditionally over $K/\mathbb{F}_q(t)$.
2. Let $X = \mathbb{P}^1$, $\phi : X \to \text{Rat}_d$ be constant, and $\beta : \mathbb{P}^1 \to \mathbb{P}^1$ be the identity map. Then we recover our previous result.

Algebraic integers of bounded degree in orbits:

1. Let $K/\mathbb{Q}$ be a number field.

Algebraic integers of bounded degree in orbits:

1. Let $K/\mathbb{Q}$ be a number field.
2. Let $\overline{\mathcal{O}}_K \subseteq \overline{K}$ be the integral closure of $\mathcal{O}_K$.

Algebraic integers of bounded degree in orbits:

1. Let $K/\mathbb{Q}$ be a number field.
2. Let $\overline{\mathcal{O}}_K \subseteq \overline{K}$ be the integral closure of $\mathcal{O}_K$.
3. For $D, B > 0$ let

$$\mathbb{P}^1(D,B,K) := \left\{P \in \mathbb{P}^1(\overline{K}) \;\middle|\; [K(P):K] \leq D,\; h(P) \leq B\right\}.$$

Algebraic integers of bounded degree in orbits:

1. Let $K/\mathbb{Q}$ be a number field.
2. Let $\overline{\mathcal{O}}_K \subseteq \overline{K}$ be the integral closure of $\mathcal{O}_K$.
3. For $D, B > 0$ let

$$\mathbb{P}^1(D, B, K) := \{P \in \mathbb{P}^1(\overline{K}) \mid [K(P) : K] \leq D,\ h(P) \leq B\}.$$

Remark: If $\phi^2 \notin \overline{K}[x]$ and $P \in \mathbb{P}^1(\overline{K})$, then

$$\mathcal{O}_\phi^+(P) \cap \overline{\mathcal{O}}_K \text{ is finite;}$$

(replace K with $K(P)$ and use Silverman's original proof)

Algebraic integers of bounded degree in orbits:

1. Let $K/\mathbb{Q}$ be a number field.
2. Let $\overline{\mathcal{O}}_K \subseteq \overline{K}$ be the integral closure of $\mathcal{O}_K$.
3. For $D, B > 0$ let

$$\mathbb{P}^1(D,B,K) := \left\{ P \in \mathbb{P}^1(\overline{K}) \mid [K(P):K] \leq D,\ h(P) \leq B \right\}.$$

Remark: If $\phi^2 \notin \overline{K}[x]$ and $P \in \mathbb{P}^1(\overline{K})$, then

$$\mathcal{O}_\phi^+(P) \cap \overline{\mathcal{O}}_K \text{ is finite;}$$

(replace K with $K(P)$ and use Silverman's original proof)

Therefore, since $\mathbb{P}^1(D,B,K)$ is finite, we can study the average value of $\#(\mathcal{O}_\phi^+(P) \cap \overline{\mathcal{O}}_K)$ for large B and fixed D.

Theorem (WH, Gunther-WH 2016)

Let $\phi \in K(x)$ *satisfy* $\phi^2 \notin \overline{K}[x]$ *and* $\deg(\phi) \geq 2$*. Then:*

1. *There exists* $N = N(\phi, K, D)$ *such that for any non-preperiodic point* Q*, we have that* $\phi^n(Q) \in \overline{\mathcal{O}}_K$ *implies* $n \leq N$*.*
2. $\#(\mathcal{O}_\phi^+(P) \cap \overline{\mathcal{O}}_K)$ *is bounded over all* $P \in \mathbb{P}^1(D, K)$*.*
3. *For all* $D > 0$*, we have that*

$$\overline{\operatorname{Avg}}(\phi, D) := \limsup_{B \to \infty} \frac{\sum_{P \in \mathbb{P}^1(D,B,K)} \#(\mathcal{O}_\phi^+(P) \cap \overline{\mathcal{O}}_K)}{\#\mathbb{P}^1(D, B, K)} = 0.$$

Theorem (WH, Gunther-WH 2016)

Let $\phi \in K(x)$ *satisfy* $\phi^2 \notin \overline{K}[x]$ *and* $\deg(\phi) \geq 2$. *Then:*

1. *There exists* $N = N(\phi, K, D)$ *such that for any non-preperiodic point* Q, *we have that* $\phi^n(Q) \in \overline{\mathcal{O}}_K$ *implies* $n \leq N$.
2. $\#(\mathcal{O}_\phi^+(P) \cap \overline{\mathcal{O}}_K)$ *is bounded over all* $P \in \mathbb{P}^1(D, K)$.
3. *For all* $D > 0$, *we have that*

$$\overline{\text{Avg}}(\phi, D) := \limsup_{B \to \infty} \frac{\sum_{P \in \mathbb{P}^1(D,B,K)} \#(\mathcal{O}_\phi^+(P) \cap \overline{\mathcal{O}}_K)}{\#\mathbb{P}^1(D, B, K)} = 0.$$

Rough interpretation:

"A random algebraic number has no integral points in its orbit."

Thank you!