Uniform boundedness for positive dimensional varieties

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joint work with Jason Bell and Dragos Ghioca

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There is a constant $C(N, d, D)$ *such that for any morphism* $f : \mathbb{P}^N \longrightarrow \mathbb{P}^N$ *of degree* $d \geq 2$ *defined over a number field* K *with* $[K : \mathbb{Q}] \leq D$*, the number of preperiodic points of* f *defined over* K *is bounded by* $C(N, d, D)$*.*

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This question is very much open, but work of Morton-Silverman, Zieve, Pezda, Benedetto, and Hutz provides bounds $C(N, d, D, \mathfrak{p})$ given a prime $\mathfrak{p}$ of good reduction for f.

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Question

For $f : \mathbb{P}^N \longrightarrow \mathbb{P}^n$ defined over a number field K, is the number of preperiodic K-subvarieties finite?

Counterexamples

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Example

(Medvedev-Scanlon) Let g be a polynomial and let $f : \mathbb{A}^2 \longrightarrow \mathbb{A}^2$ by $f(x, y) = (g(x), g(y))$ (this extends to $\mathbb{P}^2$). Then any curve of the (parametrized) form $(t, g^n(t))$ is is fixed by f.

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Example

(Ghioca-T) There are morphisms $f : \mathbb{P}^3 \longrightarrow \mathbb{P}^3$ such that there are curves V where $h_f(V) = 0$ but V is not preperiodic under f.

All is not lost

If V is a periodic subvariety of $\mathbb{P}^N$ under f, we define $\mathrm{Per}(V)$ to be its (minimal) period.

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You could also ask about for a bound on the total length of orbits of preperiodic subvarieties (where you include both the period and the preperiod).

A (very) partial result

It turns out very little of the work on uniform boundedness depends much on self-morphisms of $\mathbb{P}^N$ as opposed to self-maps of other varieties X. We are able to show the following for étale self-maps of varieties. (The full statement is cumbersome so I've simplified.)

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Theorem

(Bell-Ghioca-T) Let X be a variety and let $f : X \longrightarrow X$ be an étale morphism defined over a number field K. Suppose that f has good reduction at a prime $\mathfrak{p}$ with residue field $k_\mathfrak{p}$. Then for any K-subvariety V of X such that $V(K)$ contains a smooth point, we have

$$\mathrm{Per}(V) \leq |\bar{X}(k_\mathfrak{p})| \cdot |\,\mathrm{GL}_n(k_\mathfrak{p})| \cdot p^e$$

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- *e comes from the ramification index of $\mathfrak{p}$ over p.*

Explanation of the bound

The proof uses the $\mathfrak{p}$-adic parametrization lemma and is fairly short. The upper bound

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The shape is the same as the bounds of Morton-Silverman-Zieve-Pezda-Hutz.

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1. Let α be a smooth point in $V(K)$. For an appropriate iterate f^M (where M is bounded as above), we may apply a $\mathfrak{p}$-adic parametrization lemma to any point β in $V(K_\mathfrak{p})$ that is in the residue class U_α of α to obtain a $\mathfrak{p}$-analytic map θ_β such that

$$\theta_\beta(k) = f^{kM}(\beta).$$

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$$\theta_\beta(k) = f^{kM}(\beta).$$
2. Let G be any polynomial that vanishes on V. Then, for any $\beta \in U_\alpha \cap V$, the (rigid) $\mathfrak{p}$-adic analytic function $G \circ \theta_\beta$ vanishes at infinitely many integers if V is periodic, so $G \circ \theta_\beta$ is identically zero. Thus G vanishes on $f^M(\beta)$. Hence $f^M(\beta) \in V$. This means that $f^M(U_\alpha \cap V) \subseteq V$.

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3. Since $U_\alpha \cap V$ is Zariski dense in V, this gives $f^M(V) \subseteq V$, so the period of V divides M.

Obstacles to generalizing

The obstacle to generalizing the above to more general maps $f : X \longrightarrow X$ with good reduction at a prime $\mathfrak{p}$ is finding a "good" residue class where one can apply the $\mathfrak{p}$-adic parametrization lemma. This is a serious obstacle, as it isn't clear one should exist.

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Hutz has suggested his original proof for points might be adapted to treat varieties of any dimension.

Why? Part 1

Conjecture

(Zhang, Amerik-Campana, Medvedev-Scanlon) Let $f : X \longrightarrow X$ be defined over a number field K. Then there is some $\alpha \in X(\bar{\mathbb{Q}})$ such that $\mathrm{Orb}(\alpha)$ is Zariski dense in X unless there is a nonconstant rational map $\phi : X \longrightarrow P^1$ such that $\phi \circ f = \phi$ on a dense open subset of X

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Note that $\mathrm{Orb}(\alpha)$ is Zariski dense in X unless α is contained in a preperiodic subvariety of X. Understanding preperiodic subvarieties could be a help in proving this. (Indeed, Medvedev-Scanlon have proved among the only known results in this direction by classifying the preperiodic subvarieties of certain morphisms.)

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One approach to this is specialization: if $f : X \longrightarrow X$ is defined over $\mathbb{C}$, then it's defined over a finitely generated fields L, and we can take specializations of this field to $\bar{\mathbb{Q}}$ (technically, passing to $\bar{\mathbb{Q}}$-points on a variety with function field L) to obtain maps $f_t : X_t \longrightarrow X_t$ defined over $\bar{\mathbb{Q}}$.

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Question

("Nonpreperiodic specialization") Given a morphism $f : X \longrightarrow X$ defined over a finitely generated field L and a subvariety V of f that is not preperiodic, can one find a good specialization $f_t : X_t \longrightarrow X_t$ such that V_t is not preperiodic?

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Problem: Call-Silverman specialization is hard to use here since canonical height zero does not imply preperiodic for positive dimensional subvarieties.

Why? Part 2, continued

A good answer to the question of nonpreperiodic specialization might be obtained from a good boundedness result for periods (or maybe orbit lengths) of subvarieties. There are various examples of this already in special cases of the dynamical Mordell-Lang conjecture (e.g. recent work of Ghioca-Nguyen), but the techniques are ad hoc.

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The idea is that while there can be a Zariski dense set of t such that V_t is preperiodic, there *cannot* be a Zariski dense set of t such that V_t has period bounded by any given M.