Some applications of the Arithmetic euqidistribution theorem in Complex Dynamics

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Arithmetic euqidistribution

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Chambert-Loir showed the equidistribution of points with small height on curves, while for general varieties the equidistribution theorem was given by Yuan.

Arithmetic euqidistribution

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Let K be a number field and $f : \mathbb{P}^1(K) \to \mathbb{P}^1(K)$ be a rational function of degree greater than one.

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More generally, the preperiodic points of $f : \mathbb{P}^n(K) \to \mathbb{P}^n(K)$ equidistribute on $\mathbb{P}^n$ w. r. t. the canonical measure μ_f.

Arithmetic euqidistribution

More examples:

On the complex plane $\mathbb{C}$, parameters $c \in \mathbb{C}$ with $f(z) = z^2 + c$ postcritcially finite (PCF, i.e. the critical point 0 is preperiodic), equidistribute w. r. t. the harmonic (bifurcation) measure of the Mandelbrot set.

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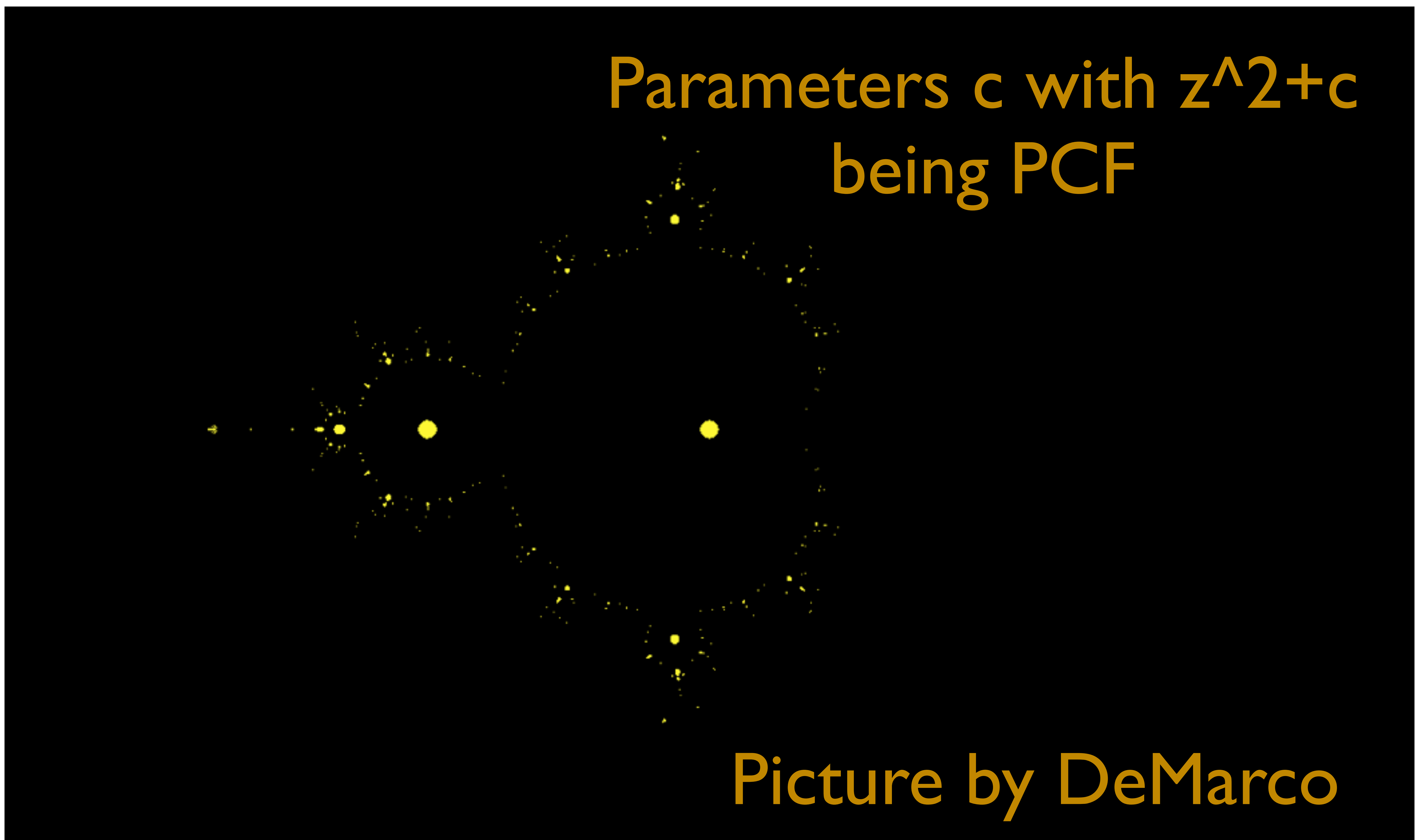

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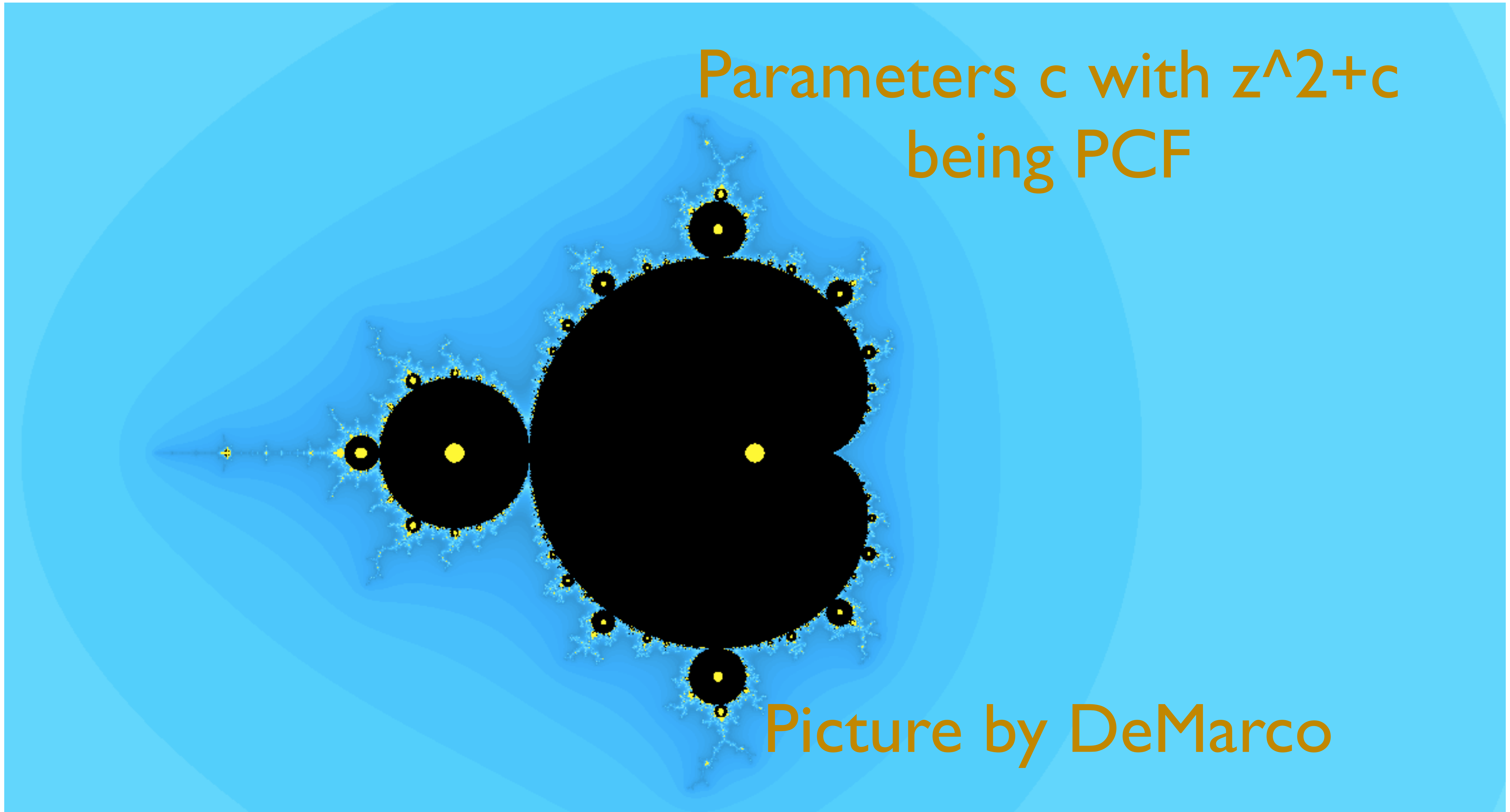

Arithmetic euqidistribution and application

More examples (Baker-DeMarco):

Fixed an $a \in K$, parameters $c \in \mathbb{C}$ with a being preperiodic under $f(z) = z^d + c$, equidistribute w.r.t. the bifurcation measure on $\mathbb{C}$ for the marked point a.

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They use this to show that if $a^d \neq b^d$, then only finitely many c with both a and b preperiodic under $f(z) = z^d + c$.

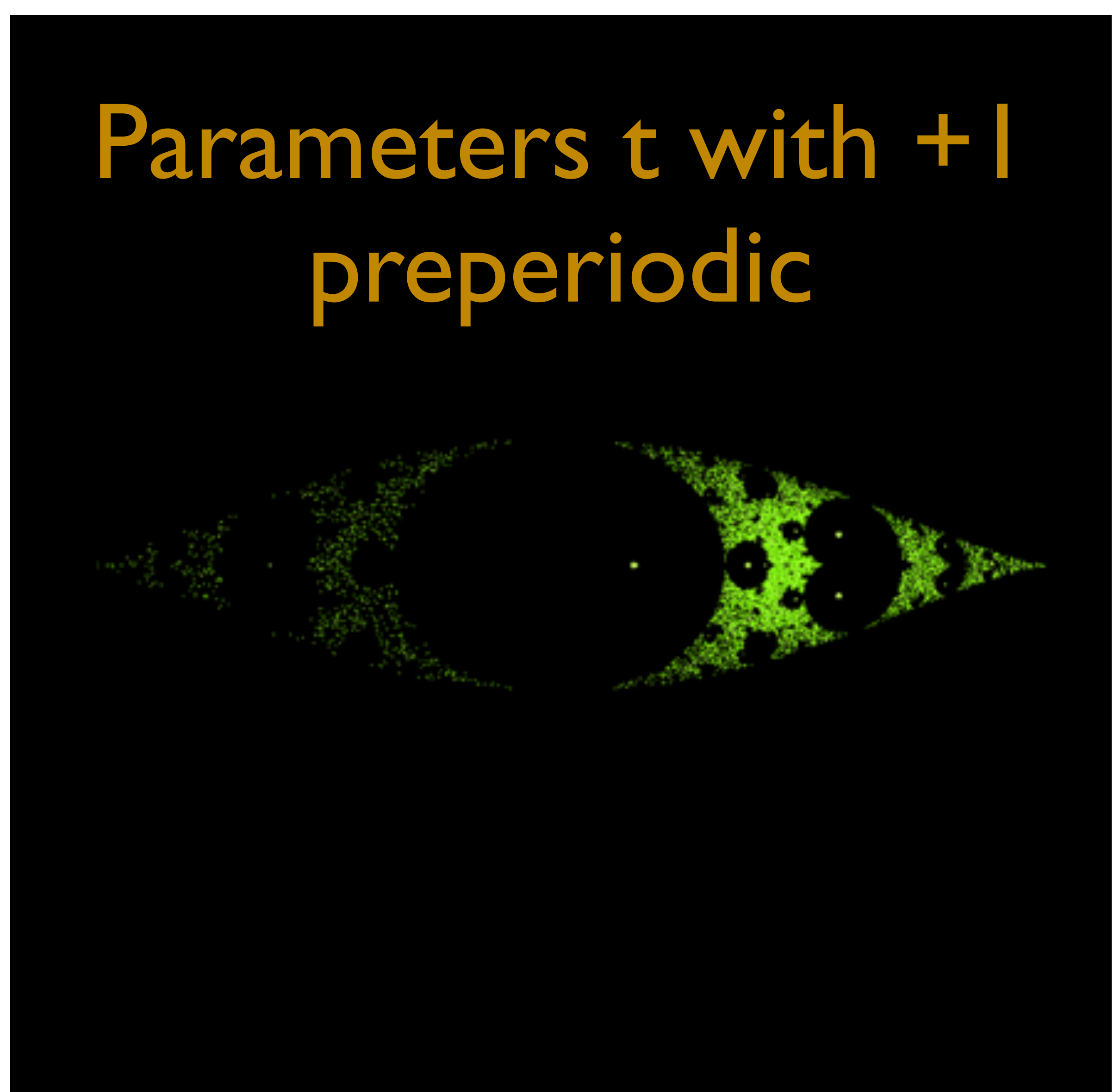
Arithmetic euqidistribution and application

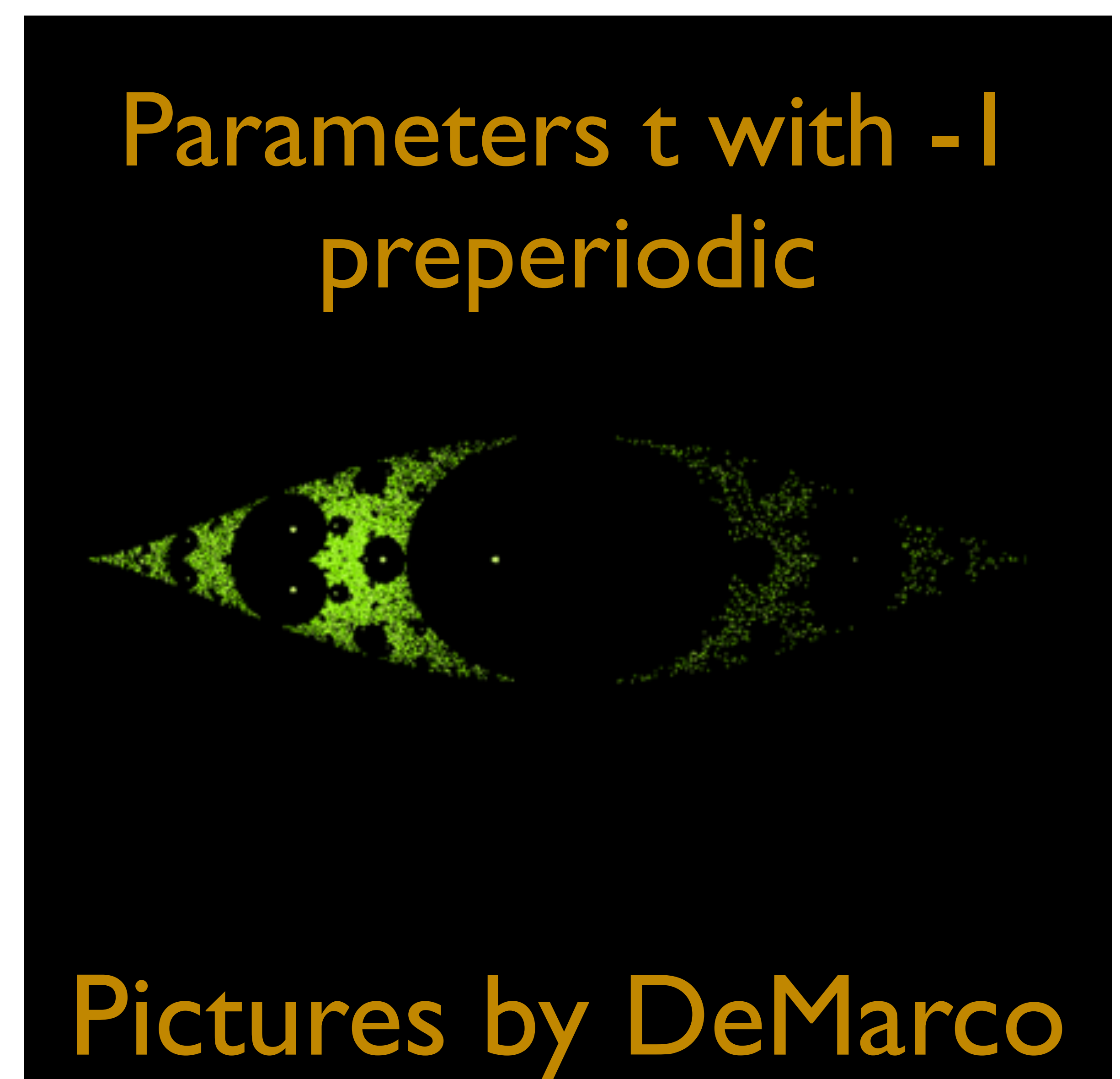
More examples (DeMarco-Wang-Y.):

Fix λ, for family

$$f_t(z) = \frac{\lambda z}{z^2 + tz + 1}$$

the set of parameter t with critical points ± 1 preperiodic, equidistributes on $\mathbb{C}$ w.r.t. the corresponding bifurcation measures.





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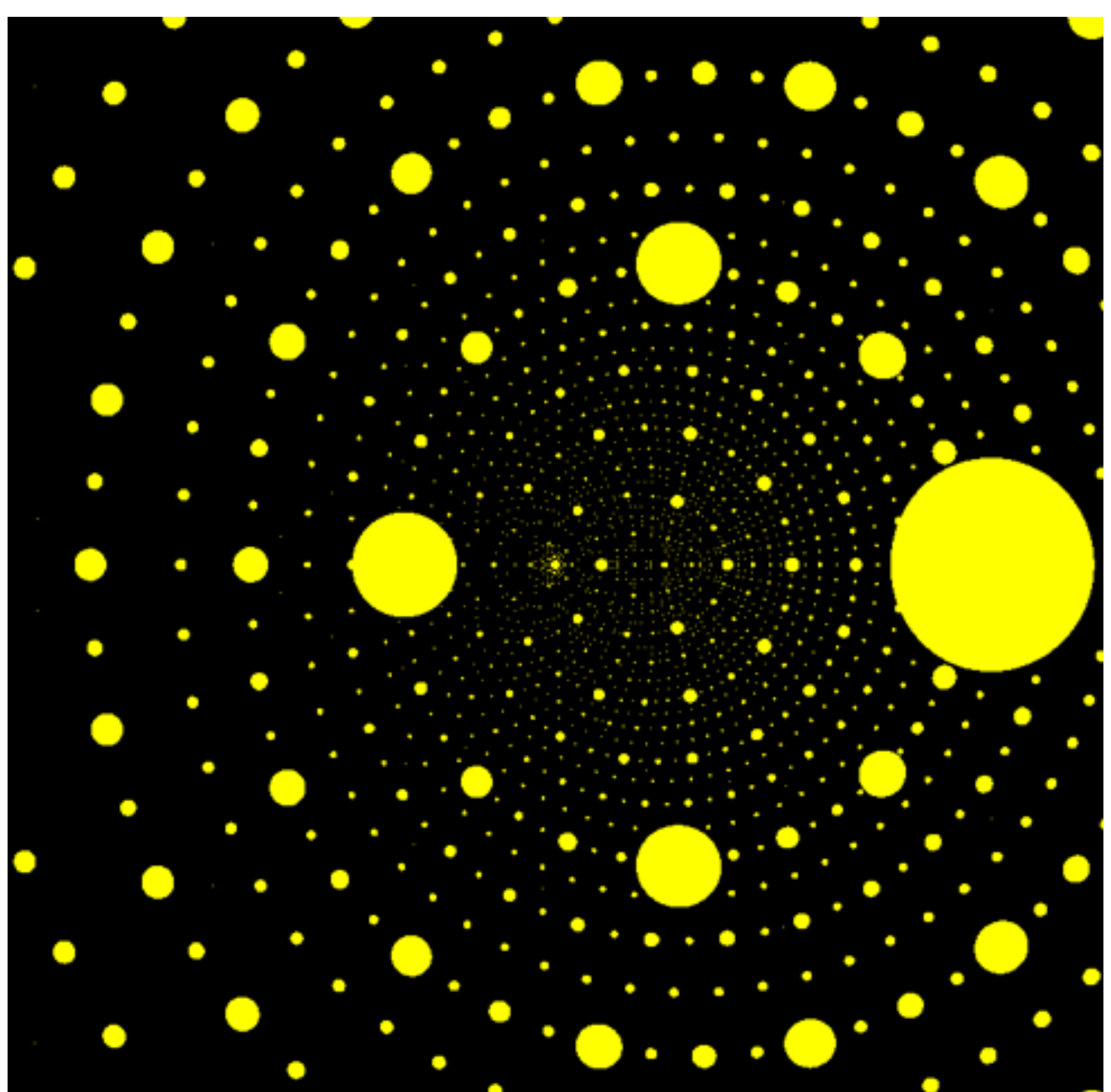
We use this to show that when $\lambda \neq 0$, there are only finitely many PCF points in $\mathrm{Per}_1(\lambda) \subset M_2$ (moduli space of quadratic maps).

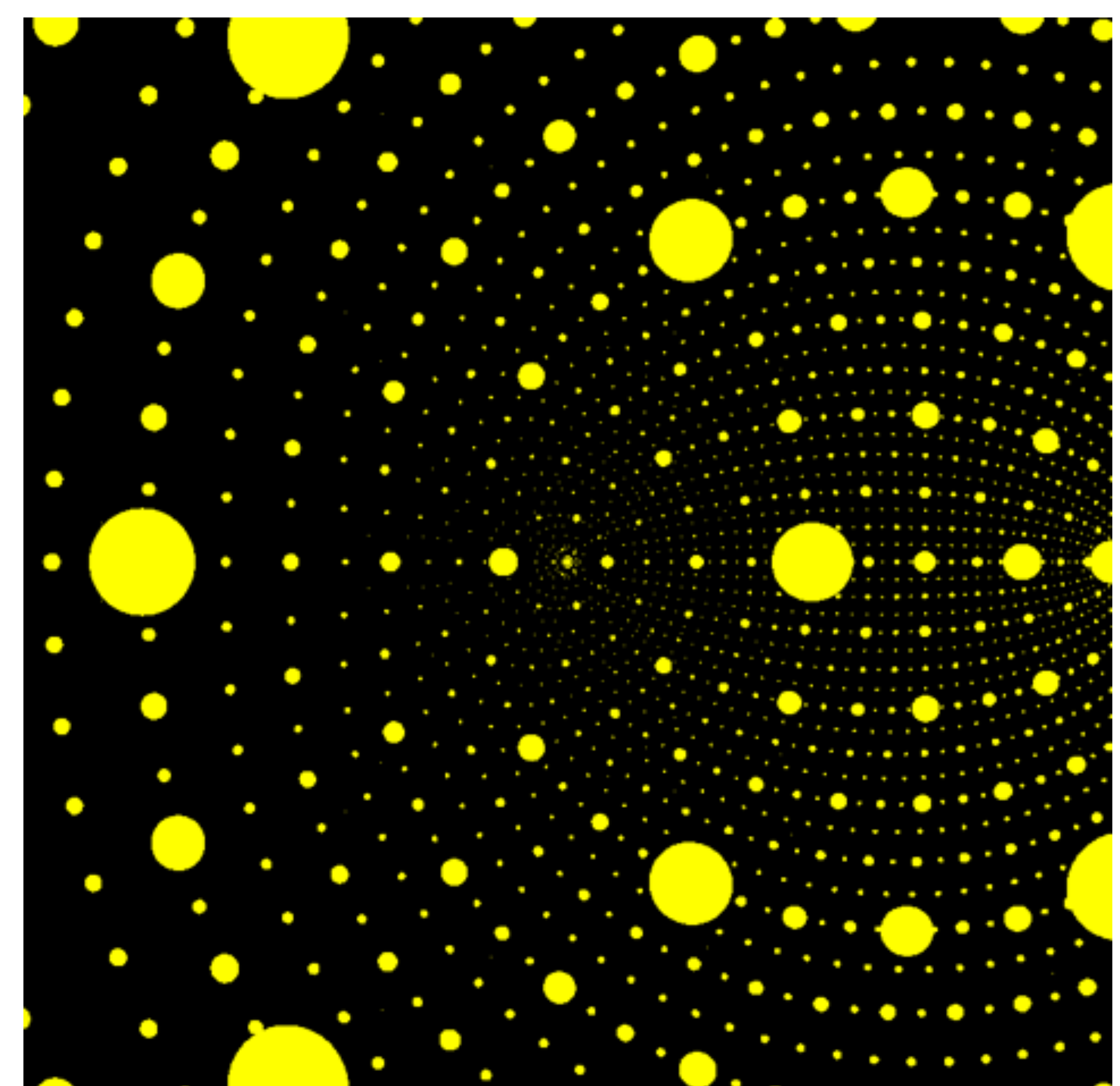
Arithmetic euqidistribution and application

More examples (DeMarco-Wang-Y.):

Legendre family $E_t : \{y^2 = x(x-1)(x-t)\}$. Take a constant section $P_a(t) = (a, \sqrt{a(a-1)(a-t)})$ ($a \in K$ a constant).

The parameters $t \in \mathbb{C}$, with $P_a(t)$ being torsion in E_t, equidistribute w.r.t. to some bifurcation measure.





Pictures by DeMarco

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Use this, we get (Originally proved by Masser and Zannier)

For $a \neq b \in K$, there are only finitely many t with both $P_a(t)$ and $P_b(t)$ torsion in E_t.

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A question asked by Baker and DeMarco:

In the moduli space of rational functions, which kind of variety contains Zariski dense PCF points?

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An analogous result (Ghioca-Krieger-Nguyen-Y.):

If non-vertical, non-horizontal curve $C \subset \mathbb{C}^2$ contains Zariski dense (a, b) with both $z^d + a$ and $z^d + b$ PCF, then C is given by $y - \eta x = 0$ with $\eta^{d-1} = 1$.

Ideas: (1) Equidistribution of small points

(2) Combinatorial properties of the Mandelbrot set

Arithmetic euqidistribution and application

Dynamical Manin-Mumford Conjecture (by Shouwu Zhang):

The relation of preperiodic subvariety and subvariety with Zariski dense preperiodic points for a dynamic system.

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We (Ghioca-Nguyen-Y.) proved this conjecture for the case when the underline dynamic space is $\mathbb{P}^1 \times \mathbb{P}^1$

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(2) Symmetries of Julia set by Levin

(3) Some other techniques

Thank you!