Height bounds and preperiodic points for certain families of polynomials

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Uniform boundedness conjecture

Let f be a rational function defined over a number field K.

Write

$$\text{Preper}(f, K) = \{P \in K : f^n(P) = f^m(P) \text{ for some } n \neq m\}.$$

Question (Morton-Silverman)

Is there a bound on $\#\,\text{Preper}(f, K)$ *depending just on* K *and* $\deg(f)$*?*

Question

Is there any family parametrized over a variety for which one can prove a uniform bound on $\#\,\text{Preper}(f, K)$*?*

Some known (and non-trivial) examples

If the family is a family of Lattés maps, then yes! [Mazur-Merel]

If the family is isotrivial (i.e., the base is morally 0-dimensional), the answer is yes! [Levy-Manes-Thomson]

If the base itself has very few K-rational points... [e.g., Faltings]

Canonical height lower bounds

Let

$$\hat{h}_f(P) = \lim_{n\to\infty} \deg f^{-n} h(f^n(P)).$$

It is natural to ask whether $\hat{h}_f(P) \neq 0$ implies

$$\hat{h}_f(P) \gg \{\text{some natural invariants of } f\}.$$

Conjecture (Silverman)

There exists an $\epsilon > 0$ depending just on $\deg(f)$ and K such that for each $P \in K$, $\hat{h}_f(P) = 0$ or

$$\hat{h}_f(P) \geq \epsilon \max\{h_{\mathsf{M}_d}(f), \log \mathrm{N}_{K/\mathbb{Q}} \mathcal{R}_f\}.$$

For $t \mapsto f_t$ over $\mathbb{P}^1$, $h(t)$ is a natural proxy for the RHS.

Some results

Theorem (Benedetto)

For f a polynomial, $\#\,\mathrm{Preper}(f,K)$ is bounded just in terms of the number of primes of bad reduction for f.

Theorem (I.)

For $f_t(z) = z^d + t$, we have $\hat{h}_{f_t}(P) = 0$ or $\hat{h}_{f_t}(P) \geq \epsilon h(t)$, where ϵ depends on the number of primes of bad reduction (actually, a potentially small subset).

See also results of Canci, Silverman, ...

Weighted homogeneous polynomials

The proof of the second result above is made convenient by the fact that $f_t(z) = z^d + t$ is a weighted-homogeneous form.

Call the family f_t *weighted homogeneous* if $f_t(z)$ is a form in z and t with z having weight 1, and t having weight $e \geq 2$, and $f_0(z), f_t(0) \neq 0$.

Theorem (I.)

For $f_t(z)$ and weighted-homogeneous family, we have $\hat{h}_{f_t}(P) = 0$ or $\hat{h}_{f_t}(P) \geq \epsilon h(t)$, where ϵ depends on the number of primes of bad reduction (actually, a potentially small subset).

Type II places

Places v where $v(t) \neq ev(P)$ are easy to handle. In particular, places where $e \nmid v(t)$ admit trivial lower bounds on the local canonical height.

If the denominator of t is far from being an eth power, then the constant in the theorem can be made absolute.

abc

Theorem (I.)

Let $f_t(z)$ be a weighted homogeneous family, let $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational function with at least N_e affine poles of order prime to e, where

$$N_e = \begin{cases} 5 & e = 2 \\ 4 & e = 3 \\ 3 & \textit{otherwise}, \end{cases}$$

and finally suppose that the abc Conjecture holds for K. Then $\hat{h}_{f_{\varphi(t)}}(P) = 0$ or

$$\hat{h}_{f_{\varphi(t)}}(P) \geq \epsilon h(\varphi(t)),$$

where $\epsilon > 0$ is independent of t.

The proof also gives a bound on $\#\,\mathrm{Preper}(f_{\varphi(t)}, K)$, uniform under *abc*.

Is *abc* necessary?

In the previous result, *abc* is used to tell us that the eth-power part of the denominator of $\varphi(t)$ does not contribute too much of the height.

To show that a point is not preperiodic, one just needs positivity of the canonical height, not some fancy lower bound.

It turns out that this comes down to showing that the denominator of t is not nearly an eth power in a much weaker sense.

NOPE

Theorem (I.)

Let $f_t(z)$ be a weighted homogeneous family, let $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational function with at least N_e affine poles of order prime to e, where N_e is as above. Then

$$\#\,\mathrm{Preper}(f_{\varphi(t)}, K)$$

is uniformly bounded.

Theorem

For $t \in \mathbb{Q}$ and

$$f_t(z) = z^d + \frac{1}{1 + t^m}$$

with ($2 \mid d$ and $m \geq 4$) or ($3 \mid d$ and $m \geq 3$), we have

$$\mathrm{Preper}(f_t, \mathbb{Q}) = \{\infty\}.$$

Thank you.