The dynamical André-Oort conjecture...

Holly Krieger

January 9, 2016
Setting: a point \((a, b) \in \mathbb{C}^2\) is **special** if each coordinate is the \(j\)-invariant of an elliptic curve with complex multiplication.

**Theorem (André 1998)**

Let \(C\) be an irreducible algebraic curve in the affine plane \(\mathbb{C}^2\). \(C\) has infinitely many special points if and only if \(C\) is either a projection fiber over a CM \(j\)-invariant, or \(C\) is a classical modular curve \(\Phi_N(x, y) = 0\).
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This is a result about (the lack of) unlikely intersections: though infinite, the special points are sparse in \(\mathbb{C}^2\), and so a plane curve only has Zariski dense special points if there's a good reason for it.

- (dynamical) André-Oort
- (dynamical) Manin-Mumford
- (dynamical) Mordell-Lang
Dynamical setting: view $\mathbb{C}^2$ as parametrizing pairs of quadratic polynomials via $(a, b) \leftrightarrow (z^2 + a, z^2 + b)$.

**Definition**

A rational map is **post-critically finite (PCF)** if all critical orbits have finite forward orbit. A point $(a, b) \in \mathbb{C}^2$ is **special** if both $z^2 + a$ and $z^2 + b$ are PCF maps.
$M := \{ c \in \mathbb{C} : \text{the critical orbit of } z^2 + c \text{ is bounded in modulus} \}$

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- If $C$ has infinitely many special points, then for all $(a, b) \in C$, $z^2 + a$ is PCF iff $z^2 + b$ is PCF
- an appropriately chosen local holomorphic branch of $\pi_1^{-1} \circ \pi^2$ induces a linear action on an interval of external angles of the Mandelbrot set
- such an action must be trivial, so $C$ is the diagonal.
Independently, Kühne and Bilu-Masser-Zannier proved effective versions of the theorem of André:

**Theorem (Kühne 2012, Bilu-Masser-Zannier 2013)**

For any irreducible algebraic curve $C$ over a number field $K$ with finitely many (CM)-special points, the set of special points is effectively computable.

Method: effectively bound the maximum of the modulus of the two discriminants for special points of $C$ which lie on no modular curve (height bound via linear forms in logs), and effectively bound the max $N$ so that a special point lies in the intersection of $C$ and $\Phi_N(x, y) = 0$. 
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**Question**

*Effective computation of the (dynamically) special points of irreducible algebraic curves in $\mathbb{C}^2$?*
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Idea: replace the modulus of the discriminant with the size of period or preperiod.

Example: (dynamically) special points on $xy = 1$. Suppose that $a$ and $1/a$ are both PCF parameters. There should exist a Galois conjugate $a^\sigma$ of $a$ which is very close to (for example) -2. However, the reciprocal $-1/2$ lies well inside the main cardioid of the Mandelbrot set and away from the hyperbolic center, so $1/a^\sigma$ cannot be PCF.
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1. **find** an element $c \in \partial M$ for which all solutions of $P(c, y) = 0$ lie inside and away from the center of hyperbolic components of the Mandelbrot set

2. **bound their period above** (and so their diameter below) by the inherent field of definition bound depending on $P$ and $c$

3. If $C$ has PCF special points whose first coordinate is PCF parameter $a$ with sufficiently large period or preperiod, then there **should be a Galois conjugate** $a^\sigma$ of that point sufficiently close to $c$ to guarantee that all solutions of $P(a^{\sigma}, y) = 0$ are non-PCF
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2. **Bound their period above** (and so their diameter below) by the inherent field of definition bound depending on $P$ and $c$. Bounds on the sizes of hyperbolic components is hard but not completely intractable.

3. If $C$ has PCF special points whose first coordinate is PCF parameter $a$ with sufficiently large period or preperiod, then there should be a Galois conjugate $a^\sigma$ of that point sufficiently close to $c$ to guarantee that all solutions of $P(a^{\sigma}, y) = 0$ are non-PCF. Effective equidistribution might do this, or analysis of external rays.
General dynamical André-Oort question?

**Definition**

Let $V$ be an irreducible quasi-projective complex variety and $d \geq 2$. $f : V \times \mathbb{P}^1 \to \mathbb{P}^1$ is an **algebraic family of rational maps** of degree $d$ if $f$ is a morphism so that $f_t := f(t, \cdot) : \mathbb{P}^1 \to \mathbb{P}^1$ is a degree $d$ morphism for each $t \in V$.

Any algebraic family of rational maps induces a projection $V \to \mathcal{M}_d$; call the dimension of the image of this projection the **dimension in moduli** of $V$.

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A **marked point** of a family $f : V \times \mathbb{P}^1 \to \mathbb{P}^1$ is a morphism $a : V \to \mathbb{P}^1$. 
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**Example:** $V = \mathbb{C}$, $f(c, [z : w]) = [z^2 + cw^2 : w^2]$, $a(c) \equiv [0 : 1]$.

**Question**

Let $V$ be an algebraic family of rational maps with marked points $a_i(t)$. For which irreducible subvarieties $Y$ of $V$ is $Y \cap S_0(V)$ Zariski dense?
<table>
<thead>
<tr>
<th><strong>Conjecture (General dynamical André-Oort conjecture, DeMarco)</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Let $f : V \times \mathbb{P}^1 \to \mathbb{P}^1$ be an algebraic family of rational maps of degree $d \geq 2$, of dimension $N &gt; 0$ in moduli. Let $a_0, ..., a_N$ be any collection of marked points. Define</td>
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<tr>
<td>$S(V, a) := \bigcap_{i=0}^{N} { t \in V : a_i(t) \text{ is preperiodic for } f_t }$.</td>
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<tr>
<td>Then $S$ is Zariski dense in $V$ if and only if the marked points are dynamically related along $V$.</td>
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<td>We say that $N$ marked points $a_1, ..., a_N$ are <strong>dynamically related</strong> along the algebraic family $f : V \times \mathbb{P}^1 \to \mathbb{P}^1$ if there exists an algebraic subvariety $X \subset (\mathbb{P}^1)^N$ defined over $\mathbb{C}(V)$ such that</td>
</tr>
<tr>
<td>1. $(a_1, ..., a_N) \in X$</td>
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<tr>
<td>2. $(f, ..., f)(X) \subset X$</td>
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<tr>
<td>3. there exists $i$ so that the projection from $X$ to the $i$th coordinate hyperplane of $(\mathbb{P}^1)^N$ is finite</td>
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How can you be sure?

Other dynamical André-Oort results:

**Theorem (Baker-DeMarco 2013)**

Let $P_3$ be the space of cubic polynomials. Given $\lambda \in \mathbb{C}$, define

$$\text{Per}_1(\lambda) := \{ f \in P_3 : f \text{ has a fixed point of multiplier } \lambda \}.$$  

Then $\text{Per}_1(\lambda)$ contains infinitely many PCF points if and only if $\lambda = 0$. 
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Theorem (DeMarco-Wang-Ye 2014)

Let $M_2$ denote the moduli space of rational maps of degree 2, modulo conjugation. Define $\text{Per}_1(\lambda)$ as above, accordingly. Then $\text{Per}_1(\lambda)$ contains infinitely many PCF maps if and only if $\lambda = 0$. 
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**Theorem (Ghioca-Hsia-Tucker 2015)**

Let $c_1, c_2, c_3$ be distinct complex numbers, and $d \geq 3$ an integer. The set of $(a_0, a_1) \in \mathbb{C}^2$ such that each $c_i$ is preperiodic for $f(z) = z^d + a_1 z + a_0$ is not Zariski dense in $\mathbb{A}^2$. 

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Galois representation of a rational map: $\phi(z)$ defined over number field $K$, $\alpha \in K$ which lies in no forward critical orbit, and consider the preimage tree $T$ of $\alpha$. The absolute Galois group $G_K$ acts on this tree, inducing a representation $\rho$ of $G_K$ to $\text{Aut}(T)$. This rep’n is generically surjective.

**Theorem (Jones-Pink)**

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Non-PCF with infinite index:

- $x^3 + 2$
- $\frac{x^2 - a}{x^2 + a}$
- $x^2 + x$ rooted at 0.
- $\frac{x^2 + 1}{x}$ rooted at 0.

Probably want an 'almost every root' or transcendental root statement.
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**Question**

Which subvarieties of $V$ contain a Zariski dense subset of arboreally special points?
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Where I end and you begin

Thanks for your attention!