The dynamical André-Oort conjecture...

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Setting: a point $(a, b) \in \mathbb{C}^2$ is **special** if each coordinate is the j-invariant of an elliptic curve with complex multiplication.

Theorem (André 1998)

Let C be an irreducible algebraic curve in the affine plane $\mathbb{C}^2$. C has infinitely many special points if and only if C is either a projection fiber over a CM j-invariant, or C is a classical modular curve $\Phi_N(x, y) = 0$.

Everything in its right place

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This is a result about (the lack of) *unlikely intersections*: though infinite, the special points are sparse in $\mathbb{C}^2$, and so a plane curve only has Zariski dense special points if there's a good reason for it.

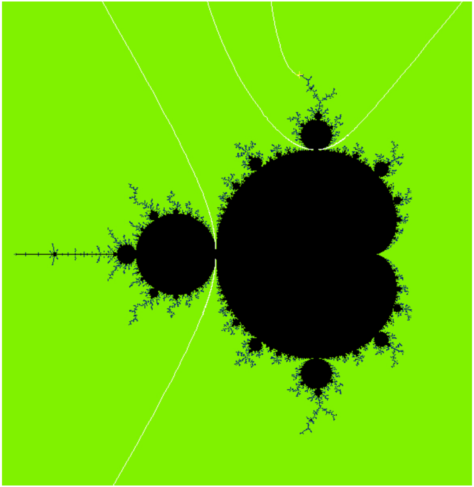
- (dynamical) André-Oort
- (dynamical) Manin-Mumford
- (dynamical) Mordell-Lang

Dynamical setting: view $\mathbb{C}^2$ as parametrizing pairs of quadratic polynomials via $(a, b) \leftrightarrow (z^2 + a, z^2 + b)$.

Definition

A rational map is **post-critically finite (PCF)** if all critical orbits have finite forward orbit. A point $(a, b) \in \mathbb{C}^2$ is **special** if both $z^2 + a$ and $z^2 + b$ are PCF maps.

$M := \{c \in \mathbb{C} : \text{ the critical orbit of } z^2 + c \text{ is bounded in modulus}\}$



Theorem (Ghioca-K.-Nguyen 2014, Ghioca-K.-Nguyen-Ye 2015)

Let C be an irreducible algebraic curve in $\mathbb{C}^2$. C has infinitely many special points if and only if C is either a projection fiber over a PCF parameter, or the diagonal.

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- If C has infinitely many special points, then for all $(a, b) \in C$, $z^2 + a$ is PCF iff $z^2 + b$ is PCF
- an appropriately chosen local holomorphic branch of $\pi_1^{-1} \circ \pi^2$ induces a linear action on an interval of external angles of the Mandelbrot set
- such an action must be trivial, so C is the diagonal.

Independently, Kühne and Bilu-Masser-Zannier proved effective versions of the theorem of André:

Theorem (Kühne 2012, Bilu-Masser-Zannier 2013)

For any irreducible algebraic curve C over a number field K with finitely many (CM)-special points, the set of special points is effectively computable.

Method: effectively bound the maximum of the modulus of the two discriminants for special points of C which lie on no modular curve (height bound via linear forms in logs), and effectively bound the max N so that a special point lies in the intersection of C and $\Phi_N(x, y) = 0$.

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Question

Effective computation of the (dynamically) special points of irreducible algebraic curves in $\mathbb{C}^2$?

(don't get any) Big Ideas

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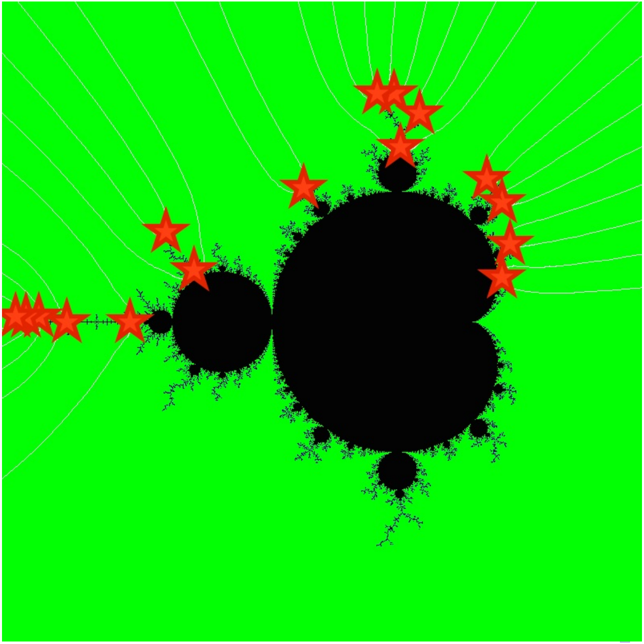
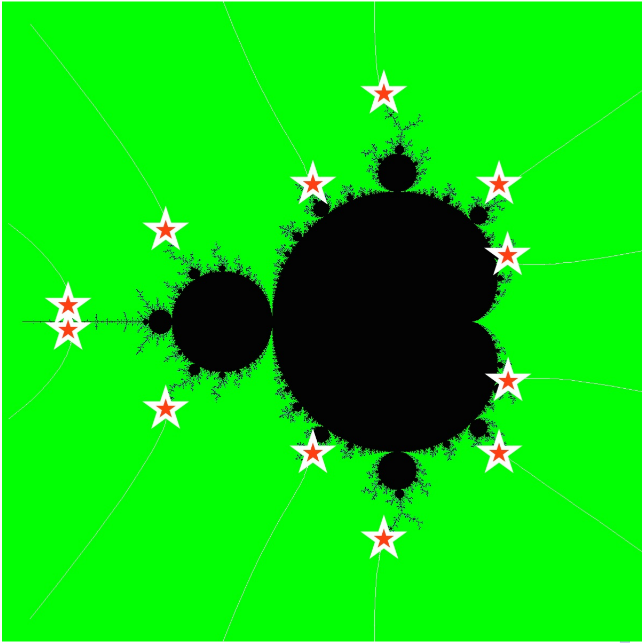
Idea: replace the modulus of the discriminant with the size of period or preperiod.

Example: (dynamically) special points on $xy = 1$. Suppose that a and $1/a$ are both PCF parameters. There should exist a Galois conjugate a^σ of a which is very close to (for example) -2. However, the reciprocal -1/2 lies well inside the main cardioid of the Mandelbrot set and away from the hyperbolic center, so $1/a^\sigma$ cannot be PCF.





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1. find an element $c \in \partial M$ for which all solutions of $P(c, y) = 0$ lie inside and away from the center of hyperbolic components of the Mandelbrot set
2. bound their period above (and so their diameter below) by the inherent field of definition bound depending on P and c
3. If C has PCF special points whose first coordinate is PCF parameter a with sufficiently large period or preperiod, then there should be a Galois conjugate a^σ of that point sufficiently close to c to guarantee that all solutions of $P(a^{sigma}, y) = 0$ are non-PCF

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Prove Yourself / I can't

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   Effective equidistribution might do this, or analysis of external rays.

Optimistic

General dynamical André-Oort question?

Definition

Let V be an irreducible quasi-projective complex variety and $d \geq 2$.
$f : V \times \mathbb{P}^1 \to \mathbb{P}^1$ is an **algebraic family of rational maps** of degree d if f is a morphism so that $f_t := f(t, \cdot) : \mathbb{P}^1 \to \mathbb{P}^1$ is a degree d morphism for each $t \in V$.

Any algebraic family of rational maps induces a projection $V \to \mathcal{M}_d$; call the dimension of the image of this projection the **dimension in moduli** of V.

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Example: $V = \mathbb{C}$, $f(c, [z : w]) = [z^2 + cw^2 : w^2]$, $a(c) \equiv [0 : 1]$.

Question

Let V be an algebraic family of rational maps with marked points $a_i(t)$. For which irreducible subvarieties Y of V is $Y \cap S_0(V)$ Zariski dense?

Conjecture (General dynamical André-Oort conjecture, DeMarco)

Let $f : V \times \mathbb{P}^1 \to \mathbb{P}^1$ be an algebraic family of rational maps of degree $d \geq 2$, of dimension $N > 0$ in moduli. Let $a_0, \ldots, a_N$ be any collection of marked points. Define

$$S(V, a) := \bigcap_{i=0}^{N} \{t \in V : a_i(t) \text{ is preperiodic for } f_t\}.$$

Then S is Zariski dense in V if and only if the marked points are dynamically related along V.

Definition

We say that N marked points $a_1, \ldots, a_N$ are **dynamically related** along the algebraic family $f : V \times \mathbb{P}^1 \to \mathbb{P}^1$ if there exists an algebraic subvariety $X \subset (\mathbb{P}^1)^N$ defined over $\mathbb{C}(V)$ such that

1. $(a_1, \ldots, a_N) \in X$
2. $(f, \ldots, f)(X) \subset X$
3. there exists i so that the projection from X to the ith coordinate hyperplane of $(\mathbb{P}^1)^N$ is finite

Other dynamical André-Oort results:

Theorem (Baker-DeMarco 2013)

Let P_3 be the space of cubic polynomials. Given $\lambda \in \mathbb{C}$, define

$$Per_1(\lambda) := \{f \in P_3 : f \text{ has a fixed point of multiplier } \lambda\}.$$

Then $Per_1(\lambda)$ contains infinitely many PCF points if and only if $\lambda = 0$.

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Theorem (DeMarco-Wang-Ye 2014)

Let M_2 denote the moduli space of rational maps of degree 2, modulo conjugation. Define $Per_1(\lambda)$ as above, accordingly. Then $Per_1(\lambda)$ contains infinitely many PCF maps if and only if $\lambda = 0$.

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Theorem (Ghioca-Hsia-Tucker 2015)

Let c_1, c_2, c_3 be distinct complex numbers, and $d \geq 3$ an integer. The set of $(a_0, a_1) \in \mathbb{C}^2$ such that each c_i is preperiodic for $f(z) = z^d + a_1 z + a_0$ is not Zariski dense in $\mathbb{A}^2$.

Galois representation of a rational map: $\phi(z)$ defined over number field K, $\alpha \in K$ which lies in no forward critical orbit, and consider the preimage tree T of α. The absolute Galois group G_K acts on this tree, inducing a representation ρ of G_K to $\mathrm{Aut}(T)$. This rep'n is generically surjective.

Theorem (Jones-Pink)

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Non-PCF with infinite index:

- $x^3 + 2$
- $\frac{x^2 - a}{x^2 + a}$
- $x^2 + x$ rooted at 0.
- $\frac{x^2+1}{x}$ rooted at 0.

Probably want an 'almost every root' or transcendental root statement.

Definition

Let V be an algebraic family of degree d dynamical systems, and call $t \in V(\bar{\mathbb{Q}})$ **arboreally special** if the arboreal representation $\rho_{t,\alpha}$ associated to $f_t(z)$ has infinite index image for almost every root α.

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Question

Which subvarieties of V contain a Zariski dense subset of arboreally special points?

Thanks for your attention!