Automorphism Groups and Invariant Theory on PN

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Action by Conjugation

Definition

Let $f : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism. For $\alpha \in \mathrm{PGL}_{N+1}$ define the conjugate of f as

$$f^\alpha = \alpha^{-1} \circ f \circ \alpha.$$

Definition

For $f : \mathbb{P}^N \to \mathbb{P}^N$ define the automorphism group of f as

$$\mathrm{Aut}(f) = \{\alpha \in \mathrm{PGL}_{N+1} \mid f^\alpha = f\}.$$

Example

Example

The map

$$f(z) = \frac{z^2 - 2z}{-2z + 1}$$

has automorphism group

$$\{z, \frac{1}{z}, \frac{z-1}{z}, \frac{1}{1-z}, \frac{z}{z-1}, 1 - z\} \cong S_3.$$

Verifying $\alpha(z) = \frac{1}{z}$ is an automorphism of f we compute

$$f^{\alpha}(z) = \frac{1}{f(1/z)} = \frac{-2/z + 1}{1/z^2 - 2/z} = \frac{-2z + z^2}{1 - 2z} = f(z).$$

Why maps with Automorphisms?

- Related to the question of field of definition versus field of moduli.
- Related to the existence of non-trivial twists (conjugate over $\overline{K}$ but not over K).
- Provides a class of morphisms with additional structure.

Theorem (Petsche, Szpiro, Tepper)

$\mathrm{Aut}(f) \subset \mathrm{PGL}_{N+1}$ *is a finite group.*

Theorem (Levy)

The set of morphisms of projective space (up to conjugation) which have a nontrivial automorphism is a finite union of proper subvarieties.

Bound on Order

Theorem (Levy)

There is a bound on the size of Aut(f) *that depends only on* deg(f) *and* N*.*

Example

For $f : \mathbb{P}^1 \to \mathbb{P}^1$ a morphism of degree $d \geq 2$

$$\# \mathrm{Aut}(f) \leq \max(2d + 2, 60).$$

Theorem (H.,de Faria)

For $f : \mathbb{P}^2 \to \mathbb{P}^2$ *a morphism of degree* $d \geq 2$

$$\# \mathrm{Aut}(f) \leq 6(d + 1)^2.$$

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Computational Problems

1. Given a finite subgroup of Γ of PGL_{N+1} is there a morphism f with $\Gamma \subseteq \mathrm{Aut}(f)$?
2. Given a morphism f, compute $\mathrm{Aut}(f)$.

Answers

Theorem (H., de Faria)

Let Γ be a finite subgroup of PGL_{N+1}. Then there are infinitely many endomorphisms $f : \mathbb{P}^N \to \mathbb{P}^N$ such that $\Gamma \subseteq \mathrm{Aut}(f)$.

Determining Aut(f)

- $f : \mathbb{P}^1 \to \mathbb{P}^1$: Faber-Manes-Viray 2014
- $f : \mathbb{P}^2 \to \mathbb{P}^2$: de Faria 2015 (MS thesis)

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Finite subgroups of PGL_2

For $f : \mathbb{P}^1 \to \mathbb{P}^1$ the conjugation action is by PGL_2. The finite subgroups of PGL_2 are

- C_n: Cyclic group of order n for $n \geq 1$.
- D_{2n}: Dihedral group of order $2n$ for $n \geq 2$.
- A_4: Tetrahedral group of order 12.
- S_4: Octahedral group of order 24.
- A_5: Icosahedral group of order 60.

All are possible for $\mathbb{P}^1$

Each of the following f have $\mathrm{Aut}(f) = \Gamma$ for each Γ.

$$C_n : f(x,y) = (x^{n+1} + xy^n : y^{n+1})$$
$$D_{2n} : f(x,y) = (y^{n-1} : x^{n-1})$$
$$A_4 : f(x,y) = (\sqrt{-3}x^2y - y^3 : x^3 + \sqrt{-3}xy^2)$$
$$S_4 : f(x,y) = (-x^5 + 5xy^4 : 5x^4y - y^5)$$
$$A_5 : f(x,y) = (-x^{11} - 66x^6y^5 + 11xy^{10} : 11x^{10}y + 66x^5y^6 - y^{11})$$

Theorem (H., de Faria)

There is no morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ defined over $\mathbb{Q}$ which has tetrahedral group as automorphism group.

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Definition

We say that F is a (relative) invariant of Γ if for all $\gamma \in \Gamma$, $\gamma F = \chi(\gamma) F$ for some linear group character χ. The set of all invariants is a ring denoted $K[\bar{x}]^{\Gamma}$. We will denote $K[\bar{x}]^{\Gamma}_{\chi}$ the ring of relative invariants associated to the character χ.

Eagon and Hochester proved that if the order of the group is relatively prime to Char K, then $K[\bar{x}]^{\Gamma}$ is Cohen-Macaulay .

Example

For $\Gamma = S_n$, then $K[\bar{x}]^{\Gamma} = \langle \sigma_1, \ldots, \sigma_n \rangle$ where σ_k is the k-th elementary symmetric polynomial.

Connection to Automorphisms:Dimension 1

Proposition (Klein 1922)

If $F \in K[x,y]^{\Gamma}$ then for $f_F = (F_y, -F_x)$ we have $\Gamma \subset \mathrm{Aut}(f)$.

In particular, there is a one-to-one correspondence between invariant 1-forms and maps with automorphisms given by

$$f_0 dx + f_1 dy \longleftrightarrow (f_1, -f_0) : \mathbb{P}^1 \to \mathbb{P}^1.$$

Theorem (Doyle-McMullen 1989)

A homogeneous 1-form θ is invariant if and only if

$$\theta = F\lambda + dG$$

where $\lambda = (xdy - ydx)/2$ and F, G are invariant homogeneous polynomials with the same character and $\deg G = \deg F + 2$.

No Tetrahedral over $\mathbb{Q}$

By Blichfeldt every invariant F of the tetrahedral group can be written as a product of powers of the following three invariants

$$\begin{aligned} t_1 &= x^4 + 2\sqrt{-3}x^2y^2 + y^4 \\ t_2 &= x^4 - 2\sqrt{-3}x^2y^2 + y^4 \\ t_3 &= xy(x^4 - y^4). \end{aligned}$$

Invariants for the octahedral group can be constructed from

$$\begin{aligned} s_1 &= xy(x^4 - y^4) = t_3 \\ s_2 &= x^8 + 14x^4y^4 + y^8 = t_1 t_2 \\ s_3 &= x^{12} - 33x^8y^4 - 33x^4y^8 + y^{12}. \end{aligned}$$

- For a map of Klein's form to be tetrahedral and defined over $\mathbb{Q}$ its invariant must be of the form $t_1^a t_2^a t_3^b$. However, this is the same as $s_2^a s_1^b$ so it will have octahedral symmetries.
- In general, if is constructed from invariants defined over $\mathbb{Q}$, then it must have octahedral symmetries.
- We also need to consider maps which come from a non-trivial F, G pair for the Doyle-McMullen construction with at least one invariant not defined over $\mathbb{Q}$.
- If one of F or G is not defined over $\mathbb{Q}$, then to end up with a map defined over $\mathbb{Q}$ we must have both not defined over $\mathbb{Q}$.

- Assume that F has a term of the form $cx^n y^m$ with $c \notin \mathbb{Q}$.
- We are constructing the coordinates of the map as $xF/2 + G_y$ and $yF/2 - G_x$.
  - In the first coordinate, we must have a monomial $\frac{cx^{n+1}y^m}{2}$ coming from $xF/2$, so G must have a term $-\frac{cx^{n+1}y^{m+1}}{2(m+1)}$ for the map to be defined over $\mathbb{Q}$.
  - Similarly, from the second coordinate we see that G has a term $\frac{cx^{n+1}y^{m+1}}{2(n+1)}$.
- Thus,

$$-\frac{c}{2(m+1)} = \frac{c}{2(n+1)}$$

and we conclude that $c = 0$, a contradiction.

□

Dimension > 1

Theorem (de Faria, H., Crass)

Define

$$dX^I = (-1)^{\sigma_I} dx_{i_1} \wedge \cdots \wedge dx_{i_n}$$

where I is the ordered set

$$\{i_1, \ldots, i_n\}, \qquad i_1 < \cdots < i_n$$

and for $\hat{i}$ the index not in I, σ_I is the sign of the permutation

$$\begin{pmatrix} 0 & 1 & \cdots & n \\ \hat{i} & i_1 & \ldots & i_n \end{pmatrix}.$$

Γ invariant n-forms

$$\phi = \sum_{\hat{i}=0}^{n} f_{\hat{i}} dX^I$$

are in 1-1 correspondence with maps $f = (f_0, \ldots, f_n)$ with $\Gamma \subset \mathrm{Aut}(f)$.

Simple Construction

1. We know that there are at least $N+1$ algebraically independent (primary) invariants for Γ, $p_0, \ldots, p_N$.
2. The $(N+1)$-form
$$dp_0 \wedge \cdots \wedge dp_N$$
is Γ-invariant.
3. Applying the previous theorem, this $(N+1)$-form corresponds to an f with $\Gamma \subseteq \mathrm{Aut}(f)$.

However, it is possible that f is the identity map as a projective map.

Equivariants

Definition

The polynomial mappings which commute with Γ are called equivariants (or sometimes covariants) and we denote them as

$$(K[V] \otimes W)^{\Gamma} = \{g \in K[V] \otimes W : g \circ \rho_V(\gamma) = \rho_W(\gamma) g\}.$$

In the language of this talk, if f is an equivariant for Γ, then $\Gamma \subseteq \mathrm{Aut}(f)$.

Module of Equivariants

$(K[V] \otimes W)^\Gamma$ is a Cohen-Macaulay module.

Proposition

For Γ finite and $N = \dim(V)$, there exist homogeneous polynomial invariants $p_1, \ldots, p_N$ such that $(K[V] \otimes W)^\Gamma$ is finitely generated as a free module over the ring $K[p_1, \ldots, p_N]$.

In particular, there exists homogeneous equivariants $g_1, \ldots, g_s$ such that

$$(K[V] \otimes W)^\Gamma = \bigoplus_{i=1}^{s} g_i K[p_1, \ldots, p_N].$$

Theorem (H., de Faria)

Let Γ be a finite subgroup of PGL_{N+1}. Then there are infinitely many endomorphisms $f : \mathbb{P}^N \to \mathbb{P}^N$ such that $\Gamma \subseteq \mathrm{Aut}(f)$.

Proof.

We can compute the number of fundamental equivariants $m \geq N + 1 \geq 2$. In particular, we claim there is at least one non-trivial equivariant f.

Assume that f is the identity map on projective space, i.e., $f = (Fx_0, \ldots, Fx_N)$ for some homogeneous polynomial F. This is an element in the module of equivariants, so that F must be an invariant of Γ. However, this equivariant is not independent of the trivial equivariant contradicting the fact that $m \geq 2$. □

Let $p_1, \ldots, p_N$ be primary invariants for Γ.

- Since the equivariants are a module over the ring $K[p_1, \ldots, p_N]$ we can form new equivariants as

$$h = \sum t_i g_i$$

  where $t_i \in K[p_1, \ldots, p_N]$, the g_i are equivariants, and the degrees $\deg(t_i g_i)$ are all the same.
- Each such map can be thought of as a point in some affine space $\mathbb{A}^{\tau}$. The identification is between the coefficients of the p_i in each t_i with the affine coordinates.
- We have $\tau \geq 1$ since we can find at least one pair of equivariants g_0, g_1 whose degrees are such that we can create a homogeneous map $t_0 g_0 + t_1 g_1$.

- Recall that the map F is a morphism if and only if the Macaulay resultant is non-zero and that the Macaulay resultant is a polynomial in the coefficients of the map (i.e., a closed condition). Thus, an open set in $\mathbb{A}^{\tau}$ corresponds to new equivariants.

□

We consider the Octahedral group. Using the fundamental equivariants

$$\begin{aligned} f_5(x,y) =& (-x^5 + 5xy^4 : 5x^4y - y^5) \\ f_{17}(x,y) =& (x^{17} - 60x^{13}y^4 + 110x^9y^8 + 212x^5y^{12} - 7xy^{16} \\ & : -7x^{16}y + 212x^{12}y^5 + 110x^8y^9 - 60x^4y^{13} + y^{17}) \end{aligned}$$

and the invariants

$$\begin{aligned} p_8 =& x^8 + 14x^4y^4 + y^8 \\ p_{12} =& x^{10}y^2 - 2x^6y^6 + x^2y^{10} = (x^5y - xy^5)^2 \end{aligned}$$

we constructed a new equivariant

$$f_{17} + 2p_{12}f_5.$$

$$f_{17} + 2p_{12}f_5 = \\ (x^{17} + 2x^{15}y^2 - 60x^{13}y^4 - 14x^{11}y^6 + 110x^9y^8 \\ + 22x^7y^{10} + 212x^5y^{12} - 10x^3y^{14} - 7xy^{16} \\ : -7x^{16}y - 10x^{14}y^3 + 212x^{12}y^5 + 22x^{10}y^7 + 110x^8y^9 \\ - 14x^6y^{11} - 60x^4y^{13} + 2x^2y^{15} + y^{17})$$

Generalizing this to

$$g_t = f_{17} + t \cdot p_{12}f_5$$

we compute the Macaulay resultant as

$$\text{Res}(g_t) = C \cdot (t-1)^6(t-4/3)^{16}$$

So for any choice of t except 1 and $4/3$, we produce an equivariant morphism for the Octahedral group.

Questions?