Generalized Dynamical systems: Preliminary Report

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Dynamical Systems

Definition

We say a pair (S, f) *is a* **(discrete) dynamical system** *when* $f : S \to S$ *is a self map.*

$$P \xrightarrow{f} f(P) \xrightarrow{f} f(f(P))$$

Definition

We define **the orbit of** P **under** f *to be*

$$\mathcal{O}_f(P) := \{P, f(P), f^2(P), \cdots, f^m(P), \cdots\}$$

where f^m *is m-th iterate of* f.

Definition (Dynamical Classification)

We say a point $P \in S$ *is* **fixed** *if* $\mathcal{O}_f(P) = \{P\}$, **periodic** *if* $f^m(P) = P$ *for some* $m \geq 1$ *and* **preperiodic** *if* $\mathcal{O}_f(P)$ *is finite.*

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Remark

We can study properties of preperiodic points by comparing information of P *and* $f(P)$*. For example, is*

$$\deg f \cdot h(P) - h(f(P))$$

bounded?

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Arithmetic Dynamics

Theorem (Northcott)

Let $\phi : \mathbb{P}^n(\overline{\mathbb{Q}}) \to \mathbb{P}^n(\overline{\mathbb{Q}})$ be a morphism of degree d. Then,

$$h(\phi(P)) > d \cdot h(P) + O(1).$$

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Northcott's theorem works only for polarizable endomorphisms, like endomorphisms on projective spaces.

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Corollary

If $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ *is a rational map with at least ont indeterminacy point, then* $\dfrac{1+\epsilon}{\deg f} h(f(P))$ *cannot be an upper bound of* $h(P)$*.*

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Theorem (Weak Version)

Let $\phi : \mathbb{A}^n(\overline{\mathbb{Q}}) \to \mathbb{A}^n(\overline{\mathbb{Q}})$ *be a polynomial map. If*

$$h(\phi(P)) > (1+\epsilon) \cdot h(P) + O(1)$$

holds for all $P \in \mathbb{A}^n(\overline{\mathbb{Q}})$, *then the set of preperiodic points is of bounded height.*

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Arithmetic Dynamics

Theorem (Silverman, L.)

Let $f_1, \cdots, f_k : \mathbb{A}^n \to \mathbb{A}^n$ be jointly regular family of polynomial maps (whose meromorphic extensions on $\mathbb{P}^n$ share no indeterminacy point.) Then

$$\sum_{i=1}^{k} \frac{1}{\deg f_i} h\big(f_i(P)\big) > (1+r)\, h(P) + O(1).$$

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Corollary

Let $M = \langle f_1, \cdots, f_k \rangle$ be a monoid generated by $f_1, \cdots, f_k$. Then

$$\mathrm{Preper}(M) := \big\{ P \in \mathbb{A}^n(\overline{\mathbb{Q}}) \mid \{f(P) \mid f \in M\} \big\}$$

is of bounded height

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Proposition

Let $M = \langle f_1, \cdots, f_k \rangle$ be a monoid generated by $f_1, \cdots, f_k$. Then

$$\mathrm{Preper}(M) \subset \bigcap_{i=1}^{k} \mathrm{Preper}(f_i)$$

is of bounded height

Comparison of two dynamical systems

Remark

Let $M = \langle f, g \rangle$ *be a monoid generated by* f, g. *Then 'M-periodic'* $(P \in \mathrm{Preper}(M))$ *is quite strong condition.*

Example

Let S *be a* $K3$*-surface with two noncommuting involutions,* ι_1, ι_2. *Then,* $\mathrm{Preper}(\iota_j) = S$ *while* $\mathrm{Preper}(\langle \iota_1, \iota_2 \rangle)$ *is of bounde height.*

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Question

Let $f, g : \mathbb{A}^n \to \mathbb{A}^n$ be polynomial maps whose meromorphic extensions share no indeterminacy point. Is there any way to study

$$\mathrm{Preper}(f) \cap \mathrm{Preper}(g)?$$

Comparison of two dynamical systems

Theorem (Baker-DeMarco, Yuan-Zhang)

Let $\phi, \psi : \mathbb{P}^n \to \mathbb{P}^n$ *be endomorphisms of degree at least* 2*. Then* $|\mathrm{Preper}(\phi) \cap \mathrm{Preper}(\psi)| = \infty$ *if and only if* $\mathrm{Preper}(\phi) = \mathrm{Preper}(\psi)$*.*

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Theorem (L.-Ye)

Let f, g be polynomial maps. If $f \circ g = g \circ f$ and $\mathrm{Preper}(f)$ is of bounded height, then $\mathrm{Preper}(f) \subset \mathrm{Preper}(g)$.

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Let f, g be polynomial maps. If $f \circ g = g \circ f$ and $\mathrm{Preper}(f)$ is of bounded height, then $\mathrm{Preper}(f) \subset \mathrm{Preper}(g)$.

Question

Is there any f, g such that $\mathrm{Preper}(f) \cap \mathrm{Preper}(g)$ is of bounded height while $\mathrm{Preper}(f), \mathrm{Preper}(g)$ are unbounded?

Dynamics Revisited

Definition

Let (S, f) be a dynamical system. We consider the monoid generated by f,

$$M_f := \langle f \rangle = \{Id, f, f^2 \cdots \}.$$

Then we can say (S, M_f) is a dynamical system.

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$$M_f := \langle f \rangle = \{Id, f, f^2 \cdots\}.$$

Then we can say (S, M_f) is a dynamical system.

Definition

We define **the orbit of P under M_f-action** *to be*

$$\mathcal{O}_{M_f}(P) := \{f(P) \mid f \in M_f\}.$$

Definition

We say a point $P \in S$ is **preperiodic** *if $\mathcal{O}_f(P)$ is finite.*

Action of Monoid

Definition

Let M be a monoid consisting of self maps on S:

$$M := \langle f_1, \cdots, f_k \rangle.$$

Then we say (S, M) is a dynamical system with several maps.

Definition

We define **the orbit of** *P to be*

$$\mathcal{O}_M(P) := \{f(P) \mid f \in M\}.$$

Definition

We say a point $P \in S$ is **preperiodic** *if $\mathcal{O}_M(P)$ is finite.*

Action of a Set of self maps

Definition

Let $\mathcal{M}$ be a set of self maps on S:

$$\mathcal{M} := \{f_i : S \to S \mid i \in I\}.$$

Then we say $(S, \mathcal{M})$ is a dynamical system.

Definition

We define **the orbit of** P *to be*

$$\mathcal{O}_{\mathcal{M}}(P) := \{f(P) \mid f \in \mathcal{M}\}.$$

Definition

We say a point $P \in S$ is **preperiodic** *if $\mathcal{O}_{\mathcal{M}}(P)$ is finite.*

Action of a Set of maps

Definition

Let$\mathcal{F}$ be a set consisting of maps from T to S:

$$\mathcal{F} := \{f : T \to S\}.$$

Then we say $(T, S, \mathcal{F})$ a dynamical system.

Definition

We define **the orbit of P under $\mathcal{F}$** *to be*

$$\mathcal{O}_{\mathcal{F}}(P) := \{f(P) \mid f \in \mathcal{F}\} \subset S.$$

Definition

We say a point $P \in T$ is **preperiodic** *if $\mathcal{O}_{\mathcal{F}}(P)$ is finite.*

Examples : Isogenies

Example

Let E_1, E_2 be elliptic curves defined over $\overline{\mathbb{Q}}$ and let $\mathcal{F} = \mathrm{Hom}(E_1, E_2)$ be the set of isogenies between E_1 and E_2. Then

$$\mathrm{Preper}(\mathcal{F}) = (E_1)_{tor}.$$

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Example

Let ϕ, ψ *be isogeneous Drinfeld module and let* $\mathcal{F} = \mathrm{Hom}(\phi, \psi)$ *be the sets of isogenies between* ϕ *and* ψ*. Then,*

$$\mathrm{Preper}(\mathcal{F}) = (\phi)_{tor}.$$

Preper(f) $\cap$ Preper(g) and Product map

Definition

Let $f, g : \mathbb{A}^n \to \mathbb{A}^n$ *be polynomial maps. We define*

$$f \times g : \mathbb{A}^n \to \mathbb{A}^{2n}, \quad P \mapsto (f(P), g(P)).$$

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$$\mathrm{Preper}(f) \cap \mathrm{Preper}(g) = \mathrm{Preper}(f \times g).$$

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Proposition

$$\mathrm{Preper}(f) \cap \mathrm{Preper}(g) = \mathrm{Preper}(f \times g).$$

Proposition

Let f, g have the same degree. Then f and g are jointly regular if $f \times g$ extends continuously on $\mathbb{P}^n$.

Application to Rational dynamical systems

Proposition

Let $(\mathbb{P}^n, \mathbb{P}^{2n}, \{F_k := f^k \times g^k\})$ *be the generalized dynamical system, and let* $G : \mathbb{P}^{2n} \to \mathbb{P}^{2n}$ *be the meromorphic extension of*

$$f \cdot g : \mathbb{A}^{2n} \to \mathbb{A}^{2n}, \quad (P, Q) \mapsto (f(P), g(Q)).$$

Then

$$\begin{tikzcd} & & \mathbb{P}^n \arrow[dl, "F_{k-1}"'] \arrow[d, "F_k"] \arrow[dr, "F_{k+1}"] & & \\ \cdots & \mathbb{P}^N \arrow[r, "G"'] & \mathbb{P}^N \arrow[r, "G"'] & \mathbb{P}^N & \cdots \end{tikzcd}$$

Application to Rational dynamical systems

Theorem

Let $f : \mathbb{A}^n \to \mathbb{A}^n$ be a regular polynomial automorphism and let f^{-1} be its inverse. Then there is an integer $1 \leq l \leq n-1$ such that

$$\deg f^l = \deg(f^{-1})^{n-l}.$$

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Theorem (Kawaguchi)

Let $\mathcal{F} := \{F_i = f^{li} \times f^{-(n-l)i} \mid i = 1, 2, \cdots\}$. Then

$$\widehat{h}(P) := \lim_{i \to \infty} \frac{1}{\deg F_i} h(F_i(P))$$

converges.

Proposition

$$\mathrm{Preper}(f) = \mathrm{Preper}(f^l) \cap \mathrm{Preper}(f^{-1}) = \mathrm{Preper}(\mathcal{F})$$

Generalization

Theorem (L., In progress)

Let $f, g : \mathbb{A}^n \to \mathbb{A}^n$ *be polynomial maps such that*

1. $\deg f = \deg g = d$,
2. *algebraically stable,*
3. *jointly regular,*
4. $h(f^2(P)) > \deg(g \circ f) h(P) - O(1)$ *and*
5. $h(g^2(P)) > \deg(f \circ g) h(P) - O(1)$.

Then the canonical height of $\mathcal{F} := \{f^i \times g^i \mid i = 1, 2, 3, \cdots\}$,

$$\widehat{h}_{\mathcal{F}}(P) := \lim_{i \to \infty} d^{-i} h((f^i \times g^i)(P))$$

is well-defined: it is equivalent to the Weil height on $\mathbb{P}^n$ *and* $\widehat{h}_{\mathcal{F}}(P) = 0$ *if and only if* $P \in \mathrm{Preper}(f) \cap \mathrm{Preper}(g)$.

General Theorem

Theorem (L., In progress)

Let F be a countable set of morphisms between projective spaces defined over a number field K:

$$\mathcal{F} := \{f_i : \mathbb{P}^n \to \mathbb{P}^N\}.$$

Suppose

$$\left| d_i^{-1} h(f_i(P)) - d_{i+1}^{-1} h(f_{i+1}(P)) \right|, \quad d_i = \deg f_i$$

converge uniformly. Then we can define the canonical height

$$\widehat{h}_{\mathcal{F}}(P) := \lim_{i \to \infty} \frac{1}{d_i} h\big(f_i(P)\big)$$

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Corollary

The set of preperiodic point is of bounded height.

Appreciation

Thanks for your hearing!