A Very Elementary Proof of a Conjecture of B. and M. Shapiro for Cubic Rational Functions

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- $K$ = field with characteristic 0
- $\overline{K}$ = algebraic closure of $K$
- $f, g \in \overline{K}(z)$ are equivalent if there exists a linear fractional transformation $\sigma \in \overline{K}$ such that $f = \sigma \circ g$. 
A Case of the B. and M. Shapiro Conjecture

Theorem (Eremenko-Gabrielov)

If $f \in \mathbb{C}(z)$ is a rational function with only real critical points, then $f$ is equivalent to a rational function with real coefficients.
**Goldberg**: There are at most

\[ \rho(d) := \frac{1}{d} \binom{2d - 2}{d - 1} \]

equivalence classes of degree \( d \) rational functions with a given set of critical points.
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Eremenko and Gabrielov: Using topological, combinatorial, and complex analytic techniques construct exactly $\rho(d)$ real rational functions with a given set of real critical points.
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• **But!** The relationship between a rational function and its critical points is purely algebraic, via the roots of the derivative.

• This leads to the following question:
Question:
Is there a truly elementary proof of the Eremenko and Gabrielov’s result?
**Corollary (Faber, T.)**

*Using only algebraic techniques we can show, if \( f \in \mathbb{C}(z) \) is a degree 3 rational function with only real critical points, then \( f \) is equivalent to a rational function with real coefficients.*
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- The quadratic case is trivial over any field.
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Note:

- The quadratic case is trivial over any field.
- If \( f \in \overline{K}(z) \) has critical points \( c_1, c_2 \in \mathbb{P}^1(K), \ c_1 \neq \infty \), then either \( f = \left( \frac{z-c_1}{z-c_2} \right)^2 \) or \( f = (z - c_1)^2 \).
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$f$ has 4 distinct critical points.
Normal form for cubic

- We begin with a normal form for cubic functions. For $u \in \overline{K} \setminus \{-1, -2\}$, define

  \[ f_u(z) = \frac{z^2(z + u)}{(2u + 3)z - (u + 2)}. \]  

- This function has the property that it fixes 0, 1, and $\infty$, and each of these three points is critical.
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**Lemma**

A cubic rational function that is critical at 0, 1, and \(\infty\) is equivalent to a unique $f_u$, and the fourth critical point is

$$
\phi(u) = -u \frac{u+2}{2u+3}.
$$
Proposition

If $f_u \in \overline{K}(z)$ is equivalent to a rational function with $K$-coefficients, then $u \in K$. 
Algebraic Condition

Definition

For a field $K$ and rational function $\phi \in K(z)$, we say $K$ is $\phi$-perfect if the map $\phi : \mathbb{P}^1(K) \to \mathbb{P}^1(K)$ is surjective.
Theorem (Faber, T.)

Let $K$ be a field of characteristic zero with algebraic closure $\overline{K}$. The following statements are equivalent:

1. Any cubic rational function $f \in \overline{K}(z)$ with $K$-rational critical points is equivalent to a rational function in $K(z)$.

2. $K$ is $\phi$-perfect, where $\phi(z) = -z \frac{z^2+2}{2z+3}$.
\[ \phi(z) = -z \frac{z+2}{2z+3} \]

(1) \(\Rightarrow\) (2).

- Take \(y \in K\). Solve the equation \(\phi(u) = y\) with \(u \in K\).
  If \(y = \infty\), then we may take \(u = -3/2\).
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- Then the function $f_u$ has $K$-rational critical points $\{0, 1, \infty, y\}$. 
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- Otherwise, choose \(u \in \bar{K}\) such that \(\phi(u) = y\).
- Then the function \(f_u\) has \(K\)-rational critical points \(\{0, 1, \infty, y\}\).
- Since \(f_u\) is equivalent to a rational function with \(K\)-coefficients, the proposition implies that \(u \in K\).
Suppose that $f$ has at least three critical points. WLOG, assume that 0, 1, and $\infty$ are among them. By the lemma, $f$ is equivalent to $f_u$ for some $u \in \bar{K}$. The remaining critical point is $\phi(u)$. By assumption, both solutions of $\phi(z) = \phi(u)$ lie in $P_1(K)$, so that $u \in K$. That is, $f$ is equivalent to a rational function with $K$-coefficients.

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**Corollary (Faber, T.)**

*Using only algebraic techniques, we can show if \( f \in \mathbb{C}(z) \) is a cubic rational function with only real critical points, then \( f \) is equivalent to a real rational function.*
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- \( \phi(-3/2) = \infty \), and if \( y \in \mathbb{R} \), then the equation \( \phi(z) = y \) is equivalent to a quadratic equation with discriminant \( 4(y^2 - y + 1) = (2y - 1)^2 + 3 > 0 \).
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- Hence \( \phi(z) = y \) has a real solution.
Question:
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Number fields are not \( \phi \)-perfect for any \( \phi \) with \( \deg(\phi) \geq 2 \). We can show this using a canonical height argument.

If \( \phi(z) = -z \frac{z+2}{2z+3} \), the field \( \mathbb{Q}_p \) is \( \phi \)-perfect iff \( p = 3 \).
Proof for $p > 3$.

- The resultant of $\phi(z) = -z \frac{z+2}{2z+3}$ is 3 $\Rightarrow$ reduced modulo $p$ to yield a quadratic function $\tilde{\phi} \in \mathbb{F}_p(z)$. 
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- Since this is a finite set, $\tilde{\phi}$ also fails to be surjective. Choose $\tilde{y} \in \mathbb{F}_p$ such that $\tilde{\phi}(z) = \tilde{y}$ has no solution in $\mathbb{F}_p$. 

By Hensel's lemma, it follows that $\phi(z) = y$ has no solution in $\mathbb{Z}_p$ for any $y \in \mathbb{Z}_p$ such that $y \equiv \tilde{y} \pmod{p}$. 

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Proof continued.

- It remains to show that \( \phi(z) = y \) has no solution in \( \mathbb{Q}_p \setminus \mathbb{Z}_p \).
- If \( \phi(x) = y \) with \( |x|_p > 1 \), then

\[
|\phi(x)|_p = |x|_p \cdot \left| \frac{1 + 2/x}{2 + 3/x} \right|_p = |x|_p > 1,
\]

which contradicts \( y \in \mathbb{Z}_p \). Hence \( \phi(z) = y \) has no solution in \( \mathbb{P}^1(\mathbb{Q}_p) \), and we have proved that \( \mathbb{Q}_p \) is not \( \phi \)-perfect.
Further Thoughts

- A general rational function of degree $d$ has $2d + 1$ free parameters (coefficients) and $2d - 2$ critical points.
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- $d = 3$: $2d - 5 = 1$. Express the remaining critical point as a function of the free parameter.
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$d = 3$: $2d - 5 = 1$. Express the remaining critical point as a function of the free parameter.

Is it possible to solve for the critical points as explicit functions of parameters for $d > 3$?
Further Thoughts continued

- Bézout’s Theorem gives an upper bound of $2^{2d-5}$ solutions for a general system of $2d - 5$ conics, while Goldberg bounds the number of distinct solutions by the smaller quantity

$$\frac{1}{d} \left( \frac{2d - 2}{d - 1} \right) \approx \frac{8}{\sqrt{\pi d^{3/2}}} 2^{2d-5}.$$
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- This suggests a substantial amount of extra structure in our system of equations, which may make progress possible.
Quartic

Example

\[ f(z) = \frac{(z^4 + az^3 + bz^2)}{cz^2 + dz + 1 + a + b - c - d} \]

where \( d = \frac{(3a^2 + 5ab + 2b^2 - 2ac - 2bc + 7a + 6b - 2c + 4)}{a + b + 1} \).

The critical points are:

\[ t_1 = -\frac{(ac + 9a + 6b - 4c + 12)}{(2c)} \]
\[ t_2 = \frac{(6a^2 + 4ab - 3ac + 9a + 2b)}{(2c)} \]
\[ t_3 = -\frac{b(2a + b - c + 3)}{c} \]
THANK YOU!