Apollonian Dynamics

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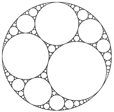
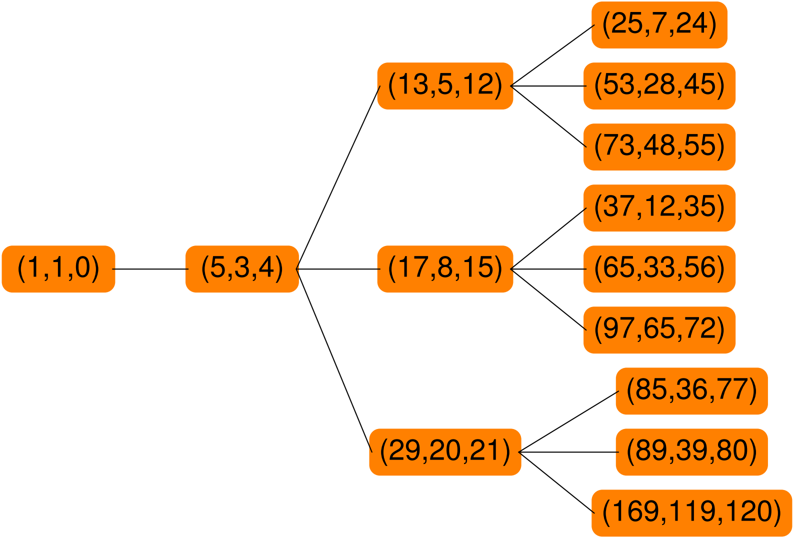


preliminary report

Joint Mathematics Meetings, January 9th, 2016

Pythagorean Triples: $\{(a, b, c) : a^2 = b^2 + c^2\}/\langle\, b \leftrightarrow c, \text{signs}\rangle$



Dynamics on a tree of Pythagorean triples (Romik, Vogeler, others?)

$$\{(a, b, c) : a^2 = b^2 + c^2, a, b, c > 0\}$$

Define a map:

$$T(a, b, c) = (3a - 2b - 2c, |2a - b - 2c|, |2a - 2b - c|)$$

Iterating this map causes you to travel through the Pythagorean tree to the origin. Note: this is *reflection along* $(1, 1, 1)$, together with sign changes.

Dynamical Statements

On the first-quadrant arc of unit circle $((b/a, c/a))$:

$$T(x, y) = \left(\frac{|2 - x - 2y|}{3 - 2x - 2y}, \frac{|2 - 2x - y|}{3 - 2x - 2y} \right).$$

Theorem (Romik)

1. *Any Pythagorean triple eventually reaches* $(1, 0)$ *or* $(0, 1)$
2. *Irrational points correspond to infinite orbits*
3. *There is an ergodic invariant measure* μ*, given by*

$$d\mu(x, y) = \frac{ds}{\sqrt{(1 - x)(1 - y)}}.$$

4. *The system is conjugate to a modified slow Euclidean algorithm*

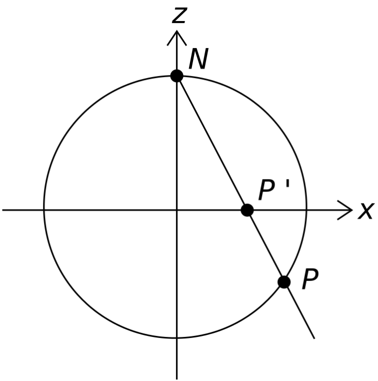
Probability consequences

There are *three* possible cases as to what T will do, in terms of sign changes, which we can record as an expansion: $1123332222332123212\ldots$

Then the following can be computed (Romik):

1. the probability of a given digit following another in random orbits
2. distribution of the n-th digit on uniformly distributed Pythagorean triples of bounded size

Stereographic projection



- The map is conjugated to a map on the unit interval, corresponding to a slow (subtractive) Euclidean algorithm.
- Related to the famous continued fraction map $\frac{1}{x} - \left\{\frac{1}{x}\right\}$.

image by Wikipedia contributor joshuardavis

Dynamics on a tree of Pythagorean triples, adjusted

$$\{(a,b,c) : a^2 = b^2 + c^2, b, c \text{ not both negative}\}$$

Define a map:

$$T(a,b,c) =$$

$$\begin{cases} (3a-2b-2c, 2a-b-2c, 2a-2b-c) & b, c > 0 \\ (3a-2b+2c, 2a-b+2c, -2a+2b-c) & b > 0, c < 0 \\ (3a+2b-2c, -2a-b+2c, 2a+2b-c) & c > 0, b < 0 \end{cases}$$

Iterating this map causes you to travel through the Pythagorean tree to the origin: it is the same system when ordering/signs are forgotten.

Reading off the path (Romik, adjusted)

Travel in the tree is by linear transformations of order 2:

$$A := \begin{pmatrix} 3 & -2 & -2 \\ 2 & -1 & -2 \\ 2 & -2 & -1 \end{pmatrix}, B := \begin{pmatrix} 3 & -2 & 2 \\ 2 & -1 & 2 \\ -2 & 2 & -1 \end{pmatrix}, C := \begin{pmatrix} 3 & 2 & -2 \\ -2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}.$$

The system 'reads off' the word giving a particular triple:

$$\delta(a,b,c) = \begin{cases} A & b,c > 0 \\ B & b > 0, c < 0 \\ C & c < 0, b > 0 \end{cases}.$$

$$M_n = \delta(T^n(a,b,c))$$

$$M_0 M_1 M_2 \cdots M_{n-1}(1,1,0) = (a,b,c)$$

Cayley graph

Let O_P be the orthogonal group preserving the form.

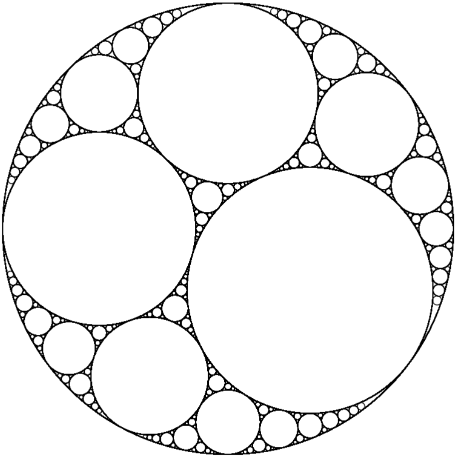
$$\Gamma = \langle A, B, C \rangle < O_P$$

So the tree is really an image of the *Cayley graph* of Γ:

vertices = Γ

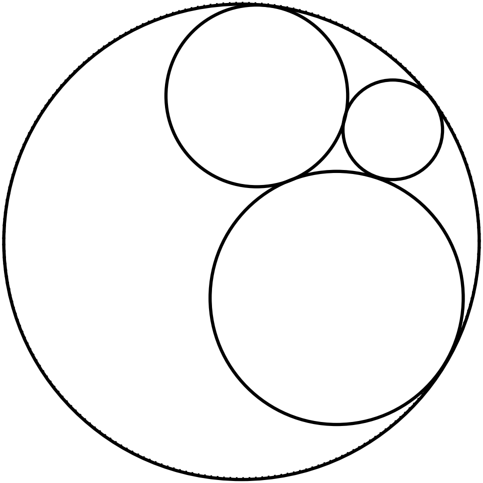
edges = if elements are related by $\times A$, $\times B$ or $\times C$

Apollonian Circle Packings



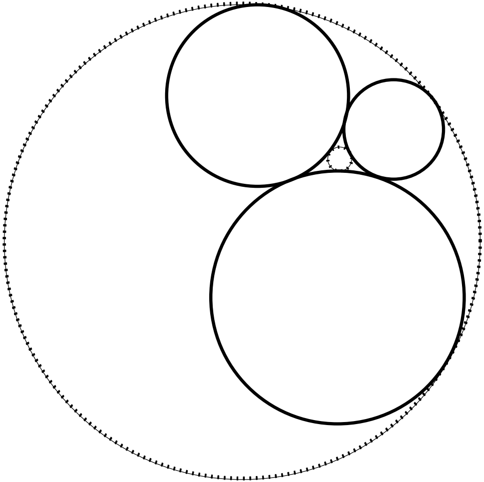
Apollonian Circle Packings

A Descartes quadruple is any collection of four circles which are pairwise mutually tangent, with disjoint interiors.



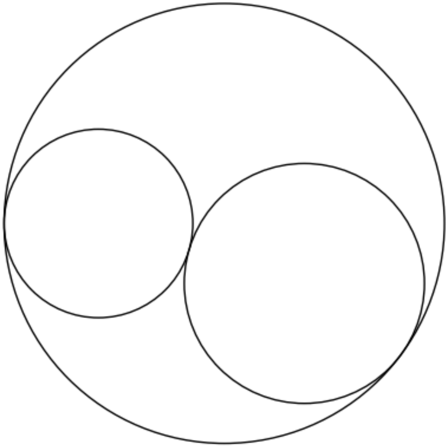
Apollonian Circle Packings

Given any three mutually tangent circles, there are exactly two ways to complete the triple to a Descartes quadruple.



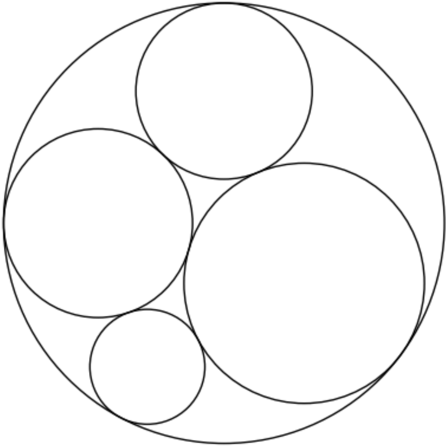
Apollonian Circle Packings

Beginning with any three mutually tangent circles...



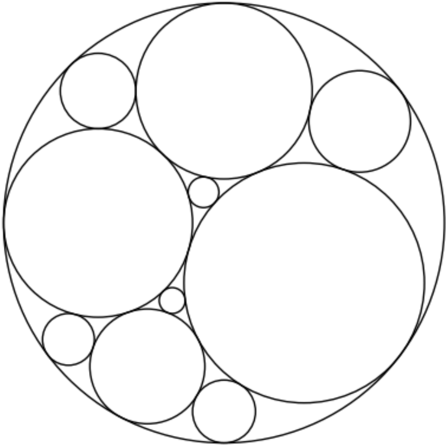
Apollonian Circle Packings

Beginning with any three mutually tangent circles, add in both new circles which would complete the triple to a Descartes quadruple.



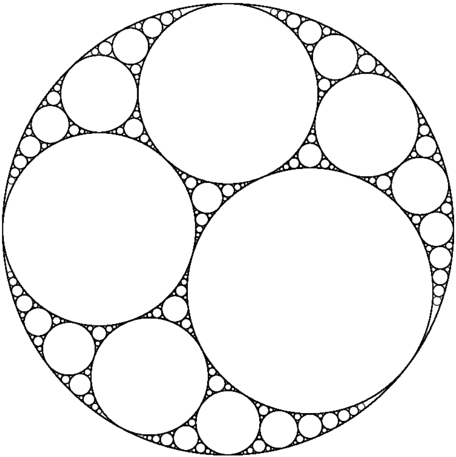
Apollonian Circle Packings

Repeat: for every triple of mutually tangent circles in the collection, add the two 'completions.'

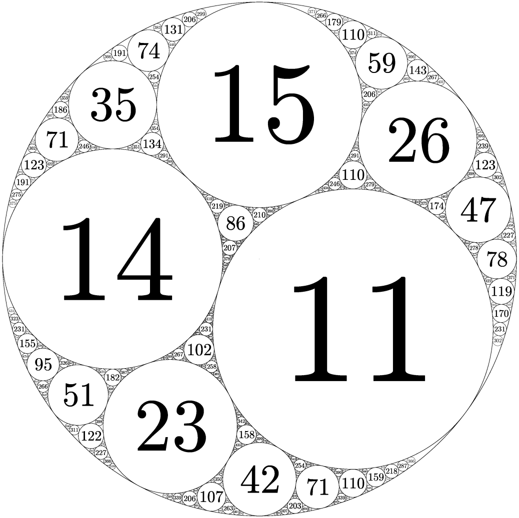


Apollonian Circle Packings

Repeating ad infinitum, we obtain an Apollonian circle packing.



Apollonian Circle Packings



The outer circle has curvature -6 (its interior is outside).

The Descartes Rule

The curvatures (inverse radii) in a Descartes configuration satisfy

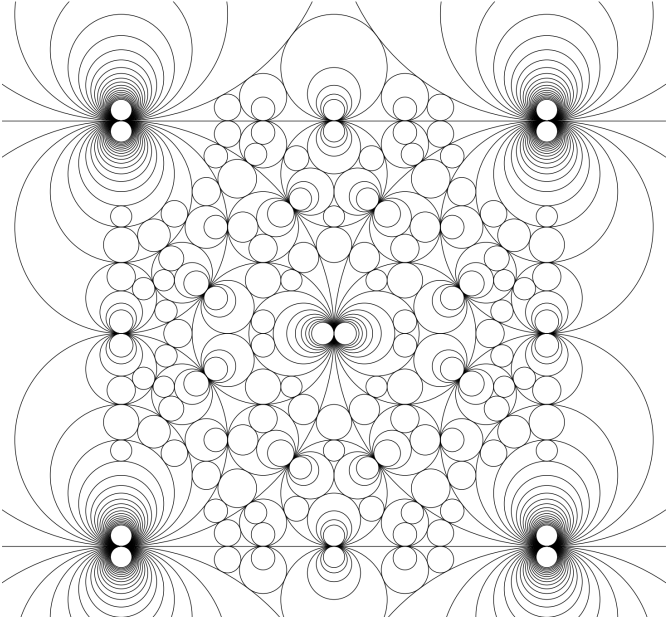
$$2(a^2 + b^2 + c^2 + d^2) = (a + b + c + d)^2.$$

If a, b, c are fixed, there are two solutions d, d', where

$$d + d' = 2(a + b + c).$$

Hence an **integer** Descartes quadruple generates an Apollonian packing of **integer curvatures.**

Schmidt Arrangement of $\mathbb{Q}(i)$



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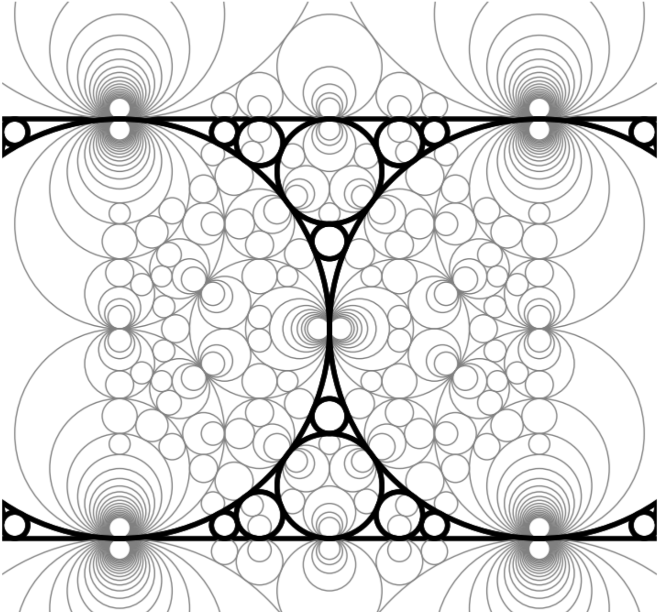
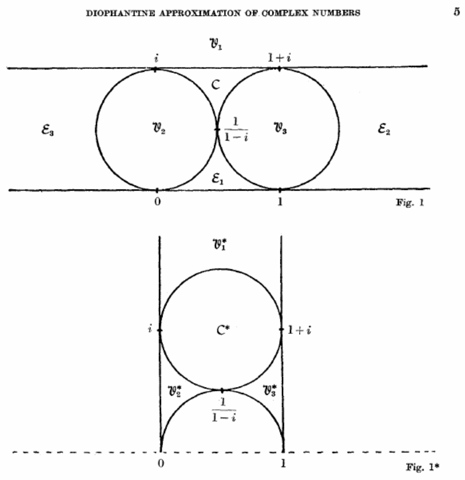


History: 1975



- Asmus Schmidt, *Diophantine Approximation of Complex Numbers*, Acta Arithmetica, 1975.
- Continued fractions for $\mathbb{Z}[i]$, $\mathbb{Z}[\sqrt{-2}]$ etc.

History: 2006

Apollonian Circle Packings, II

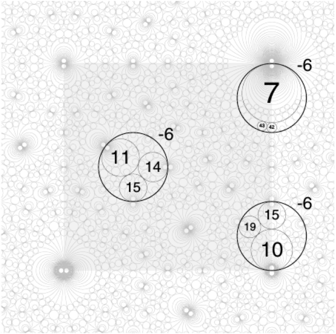
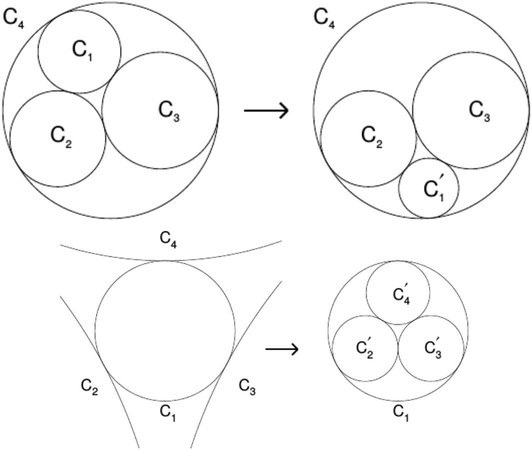


Fig. 5. Integer Apollonian packings with a bounding circle of curvature 6.

- Orbit of *super-Apollonian group*.
- Graham, Lagarias, Mallows, Wilks, Yan, *Apollonian Circle Packings: Geometry and Group Theory II.*, Discrete and Computational Geometry, 2006.

Quadruple Operations



Images: Graham, Lagarias, Mallows, Wilks, Yan, *Apollonian Circle Packings: Geometry and Group Theory I. The Apollonian Group*

Apollonian Dynamics

Replace the group $\langle A, B, C\rangle$ with the *super-Apollonian group*,

$$\langle S_1, S_2, S_3, S_4, S_1^{\perp}, S_2^{\perp}, S_3^{\perp}, S_4^{\perp}\rangle,$$

acting on quadruples of curvatures.

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acting on quadruples of curvatures.

- graph of quadruples comes from Cayley graph
- not a tree
- choose a spanning tree
- define dynamical system on spanning tree
- Question: distribution of ‘digits’ in expansion

Lorentz form

The Descartes form

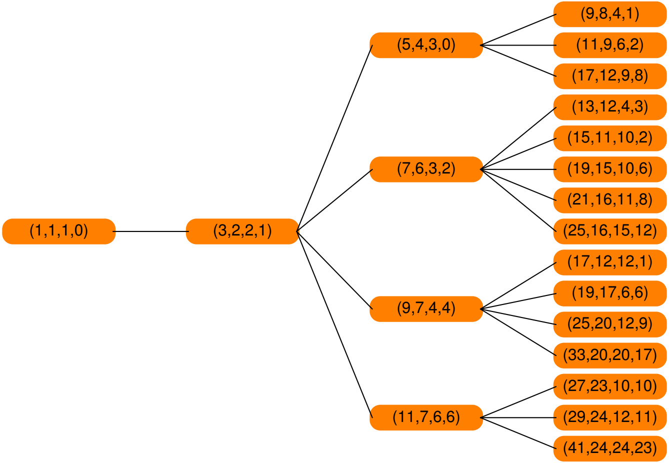
$$2(a^2 + b^2 + c^2 + d^2) = (a + b + c + d)^2$$

is conjugate to the Lorentz form

$$a^2 = b^2 + c^2 + d^2.$$

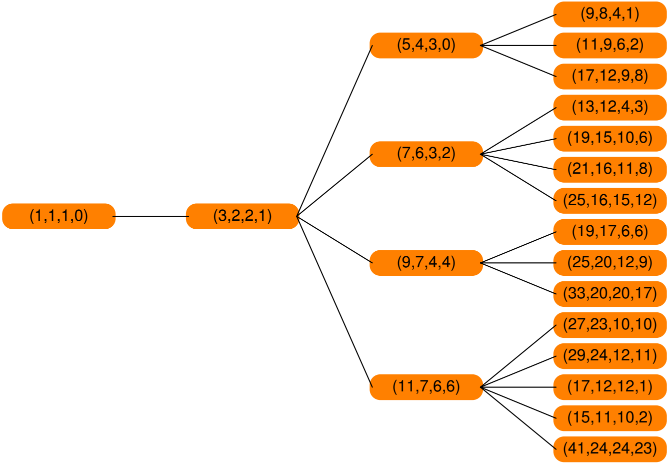
Drop down tree

Descend to origin, by repeatedly moving to the quadruple whose Lorentz form (a, b, c, d) has smallest a.



Swap down tree

Descend the Cayley graph using *normal form* (Graham et al.): No S_i^2, $(S_i^\perp)^2$ or $S_i^\perp S_j$. In other words, descend by a swap instead of an inversion if possible.



Swap down dynamical system

Define a dynamical system on Lorentz quadruples:

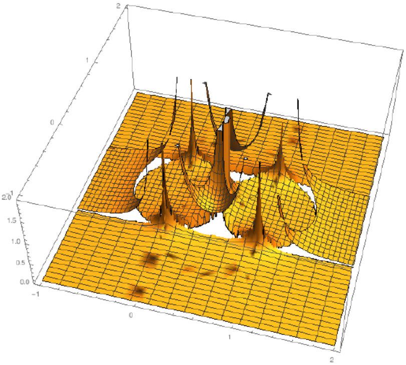
$$T(a,b,c,d) = \begin{cases} (2a-b-c-d, a-c-d, a-b-d, a-b-c) & 2a-b-c-d<0 \\ (2a-b+c+d, a+c+d, -a+b-d, -a+b-c) & 2a-b+c+d<0 \\ (2a+b+c-d, -a-c+d, -a-b+d, a+b+c) & 2a+b+c-d<0 \\ (2a+b-c+d, -a+c-d, a+b+d, -a-b-c) & 2a+b-c+d<0 \\ (2a-b-c+d, a-c+d, a-b+d, -a+b+c) & 2a-b-c+d<0 \\ (2a+b-c-d, -a+c+d, a+b-d, a+b-c) & 2a+b-c-d<0 \\ (2a-b+c-d, a+c-d, -a+b+d, a-b+c) & 2a-b+c-d<0 \\ (2a+b+c+d, -a-c-d, -a-b-d, -a-b-c) & 2a+b+c+d<0 \end{cases}$$

$$\delta(a,b,c,d) = \begin{cases} S_1 & 2a-b-c-d<0 \\ S_2 & 2a-b+c+d<0 \\ S_3 & 2a+b+c-d<0 \\ S_4 & 2a+b-c+d<0 \\ S_1^\perp & 2a-b-c+d<0 \\ S_2^\perp & 2a+b-c-d<0 \\ S_3^\perp & 2a-b+c-d<0 \\ S_4^\perp & 2a+b+c+d<0 \end{cases}$$

Theorem (Chaubey-Fuchs-Hines-S.)

$T^{(n)}(a,b,c,d) = \sigma(1,1,0,0)$ for some integer n, and some σ reordering and changing sign. Write $\delta_n = \delta(T^{(n)}(a,b,c,d))$. Then the word $\delta_{n-1}\cdots\delta_0$ is the swap normal form word for an M such that $M(a,b,c,d) = \sigma(1,1,0,0)$.

Invariant Measure



Theorem (Chaubey-Fuchs-Hines-S.)

There is an explicit invariant measure for T.

Invariant Measure

Theorem (Chaubey-Fuchs-Hines-S.)

There is an explicit invariant measure for T.

- Idea: extend T so that it is invertible (instead of $4/7$-to-1). (e.g. Keane, *A continued fraction titbit*)
- Consider $\widehat{\mathbb{C}} \times \widehat{\mathbb{C}} \setminus \Delta$, the space of oriented geodesics of hyperbolic 3-space ($(z, w) = (u + vi, x + iy)$).
- This has an isometry invariant measure

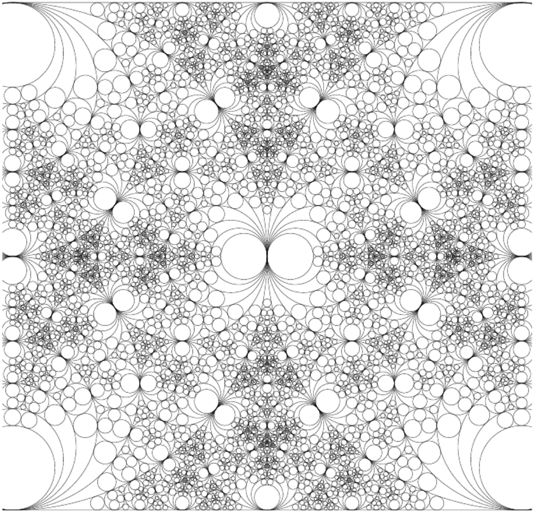
$$\frac{du\, dv\, dx\, dy}{|z - w|^4}$$

- Restrict to a subset consisting of geodesics between certain basic regions.

To Do

- Show the measure is ergodic
- Compute probabilities

Words of length ≤ 5 in swap down tree



Circles generated by normal form words of length ≤ 5.