Uniform Bounds for Periods of Endomorphisms of Varieties

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Outline





3 Some Ingredients of the Proof

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Morton and Silverman proposed the following

Conjecture (The Dynamical Uniform Boundedness Conjecture, [MS94])

Let K/\mathbb{Q} be a number field of degree D, let $\phi : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism of degree $d \ge 2$ defined over K, and let $\operatorname{Prep}(\phi, K)$ be the set of K-rational points preperiodic under ϕ . There is a constant C(D, N, d) such that

 $\#\operatorname{Prep}(\phi, K) \le C(D, N, d).$

A Variant

If we replace \mathbb{P}^N by elliptic curves,

then the above conjecture becomes the Mazur-Merel Theorem.

Theorem (Mazur-Merel)

For all $D \in \mathbb{Z}$, $D \ge 1$ there exists a constant $B(D) \ge 0$ such that for all elliptic curves E over a number field K with [K : Q] = Dwe have $|E(K)_{\text{tors}}| \le B(D)$.

Previous Results on Period

- Pezda and Zieve proved bounds for the length of integral cycles of certain polynomial endomorphisms of affine spaces.
- Fakhurddin proved a boundedness result for endomorphisms of certain proper schemes.
- Hutz proved a bound for endomorphisms of smooth projective varieties with good reduction.
- Bell, Ghioca, and Tucker proved a bound for étale morphisms of smooth models of varieties.

Hutz's Result

Theorem ([Hut09])

Let X/\mathbb{Q} be a smooth irreducible projective variety of dimension d and $f: X \to X$ a morphism defined over \mathbb{Q} with good reduction at a prime p and denote by \tilde{X} its reduction. Let $P \in X(\mathbb{Q})$ be a periodic point with primitive period n. Then we have

$$n \leq |\tilde{X}(\mathbb{F}_p)| \cdot p(p^d - 1),$$
 for $p \neq 2$

Hutz's Result

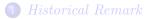
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In the case when K is a number field, there is another factor depending only on K and the prime.

Outline





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Our Main Result

Theorem (H)

Let X/\mathbb{Q} be a variety. Suppose f admits a weak notion of good reduction at a prime p and denote by \tilde{X} its reduction at p. Let $P \in X(\mathbb{Q})$ be a periodic point under f.

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Let X/\mathbb{Q} be a variety. Suppose f admits a weak notion of good reduction at a prime p and denote by \tilde{X} its reduction at p. Let $P \in X(\mathbb{Q})$ be a periodic point under f. Then the primitive period n of P satisfies that

$$n \le |\tilde{X}(\mathbb{F}_p)| \cdot p\left(p^{d'} - 1\right)$$

for $p \neq 2$, where d' is the maximum dimension of the cotangent spaces at points in $\tilde{X}(\mathbb{F}_p)$.

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In the case when K is a number field and the reduction is good, our result is weaker than the result of Hutz.

About the Weak Notion of Reduction

- There is a model \mathcal{X} of X over \mathbb{Z}_p .
- The variety X does not have to be nonsingular.
- The special fiber \tilde{X} does not have be nonsingular.
- The special fiber \tilde{X} does not even have to be irreducible.

The Setup of the Proof

We follow the proof in [Fak01] and other papers. Replacing f by an iterate we may assume that the reduction \tilde{P} of P is fixed under f. We follow the proof in [Fak01] and other papers.

Replacing f by an iterate

we may assume that the reduction \tilde{P} of P is fixed under f.

Let $\operatorname{Spec}(A)$ be the reduced subscheme of the model \mathcal{X} determined by the orbit of P.

Then f induces an \mathbb{Z}_p -automorphism σ of A.

It can be shown that \boldsymbol{A} is a local ring and

let \mathfrak{m} be the maximal ideal of A.

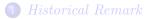
An Example

Let $K = \mathbb{Q}$, p = 3 and $X = \mathbb{P}^1$. Suppose $f : X \to X$ is given by $f(x) = x^2 - 4x + 3$ and P = 0. Then P is of primitive period 2 with orbit $O_f(P) = \{0, 3\}$.

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The Framework

We first recall Proposition 1 of [Fak01]. We will use the notations there.

Proposition

With the notations as before, we have $n \leq n_0 r p^t$ where n_0 is the primitive period of the reduction \tilde{P} of P, r is the order of the induced map on the cotangent space $\mathfrak{m}/\mathfrak{m}^2$, and t depends only on K and the prime.

Clearly $n_0 \leq |\tilde{\mathcal{X}}(k)|$.

Notations

Recall that r is the order of the induced map on $\mathfrak{m}/\mathfrak{m}^2$. First we bound r.

We can show that $\dim_k(\mathfrak{m}/\mathfrak{m}^2) \leq d'$. By a result of Darafsheh, we have $r \leq p^{d'} - 1$.

The Setup

Replacing f by an iterate we may assume that the reduction \tilde{P} of P is fixed under f.

Let $\operatorname{Spec}(A)$ be the reduced subscheme of ${\mathcal X}$ determined by the orbit of P. Then f induces an $R\text{-morphism }\sigma$ of A.

Also let \mathfrak{m} be the maximal ideal of A.

Suppose the induced map $\tilde{\sigma}$ on $\mathfrak{m}/\mathfrak{m}^2$ is the identity.

The Setup

As in [Fak01], we look at the induced map $\sigma : A \to A$. Write $\sigma = id + h$. Then $h(\mathfrak{m}) \subseteq \mathfrak{m}^2$. Let $\nu : A \setminus \{0\} \to \mathbb{Z}$ be defined as follows: for $0 \neq a \in A$, let $\nu(a)$ be the largest integer ℓ such that $a \in \mathfrak{m}^{\ell}$.

The Setup

As in [Fak01], we look at the induced map $\sigma : A \to A$. Write $\sigma = \mathrm{id} + h$. Then $h(\mathfrak{m}) \subseteq \mathfrak{m}^2$. Let $\nu : A \setminus \{0\} \to \mathbb{Z}$ be defined as follows: for $0 \neq a \in A$, let $\nu(a)$ be the largest integer ℓ such that $a \in \mathfrak{m}^{\ell}$. Since $h(\mathfrak{m}) \subseteq \mathfrak{m}^2$, for all $a \in \mathfrak{m}$ either h(a) = 0 or $\nu(h^j(a)) > \nu(a)(j > 0)$. We will show that the order of σ is a power of p.

Poonen's Method

Suppose a is of period s and we have

$$a = \sigma^s(a) = (\mathrm{id} + h)^s(a)$$

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= $a + {\binom{s}{1}}h(a) + {\binom{s}{2}}h^{2}(a) + \dots + {\binom{s}{s-1}}h^{s-1}(a) + h^{s}(a),$
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Recall that our goal is to show that s is a power of p. It suffices to show that either s = 1 or p|s. Suppose a is of period s and we have

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Recall that our goal is to show that s is a power of p. It suffices to show that either s = 1 or p|s. Suppose $s \neq 1$ and p does not divide s. Since $\nu(h^N(a)) > \nu(a)$ for $a \in \mathfrak{m}, N \ge 1$, we must have $0 \notin \mathfrak{m}^{\nu(h(a))+1}$. Contradiction!

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Thank you!

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