Uniform Bounds for Periods of Endomorphisms of Varieties

Keping Huang

University of Rochester

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Outline

1. Historical Remark
2. Our Result
3. Some Ingredients of the Proof

Morton and Silverman proposed the following

Conjecture (The Dynamical Uniform Boundedness Conjecture, [MS94])

Let $K/\mathbb{Q}$ be a number field of degree D,
let $\phi : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism of degree $d \geq 2$ defined over K,
and let $\mathrm{Prep}(\phi, K)$ be the set of K-rational points preperiodic under ϕ.
There is a constant $C(D, N, d)$ such that

$$\#\mathrm{Prep}(\phi, K) \leq C(D, N, d).$$

A Variant

If we replace $\mathbb{P}^N$ by elliptic curves,
then the above conjecture becomes the Mazur-Merel Theorem.

Theorem (Mazur-Merel)

For all $D \in \mathbb{Z}$, $D \geq 1$ there exists a constant $B(D) \geq 0$ such that for all elliptic curves E over a number field K with $[K : Q] = D$ we have $|E(K)_{\text{tors}}| \leq B(D)$.

Previous Results on Period

- Pezda and Zieve proved bounds for the length of integral cycles of certain polynomial endomorphisms of affine spaces.
- Fakhurddin proved a boundedness result for endomorphisms of certain proper schemes.
- Hutz proved a bound for endomorphisms of smooth projective varieties with good reduction.
- Bell, Ghioca, and Tucker proved a bound for étale morphisms of smooth models of varieties.

Hutz's Result

Theorem ([Hut09])

Let $X/\mathbb{Q}$ be a smooth irreducible projective variety of dimension d and $f : X \to X$ a morphism defined over $\mathbb{Q}$ with good reduction at a prime p and denote by $\tilde{X}$ its reduction. Let $P \in X(\mathbb{Q})$ be a periodic point with primitive period n. Then we have

$$n \leq |\tilde{X}(\mathbb{F}_p)| \cdot p(p^d - 1), \text{ for } p \neq 2$$

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In the case when K is a number field, there is another factor depending only on K and the prime.

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Our Main Result

Theorem (H)

Let $X/\mathbb{Q}$ be a variety.
Suppose f admits a weak notion of good reduction at a prime p and denote by $\tilde{X}$ its reduction at p.
Let $P \in X(\mathbb{Q})$ be a periodic point under f.

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Then the primitive period n of P satisfies that

$$n \leq |\tilde{X}(\mathbb{F}_p)| \cdot p\left(p^{d'} - 1\right)$$

for $p \neq 2$, where d' is the maximum dimension of the cotangent spaces at points in $\tilde{X}(\mathbb{F}_p)$.

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In the case when K is a number field and the reduction is good, our result is weaker than the result of Hutz.

About the Weak Notion of Reduction

- There is a model $\mathcal{X}$ of X over $\mathbb{Z}_p$.
- The variety X does not have to be nonsingular.
- The special fiber $\tilde{X}$ does not have be nonsingular.
- The special fiber $\tilde{X}$ does not even have to be irreducible.

The Setup of the Proof

We follow the proof in [Fak01] and other papers.
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we may assume that the reduction $\tilde{P}$ of P is fixed under f.
Let $\mathrm{Spec}(A)$ be the reduced subscheme of the model $\mathcal{X}$ determined by the orbit of P.
Then f induces an $\mathbb{Z}_p$-automorphism σ of A.
It can be shown that A is a local ring and
let $\mathfrak{m}$ be the maximal ideal of A.

An Example

Let $K = \mathbb{Q}, p = 3$ and $X = \mathbb{P}^1$.
Suppose $f : X \to X$ is given by $f(x) = x^2 - 4x + 3$ and $P = 0$.
Then P is of primitive period 2 with orbit $O_f(P) = \{0, 3\}$.

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Then P is of primitive period 2 with orbit $O_f(P) = \{0, 3\}$.
In this case the ring $A = \mathbb{Z}_3[x]/x(x-3)$.
It's a local ring with maximal ideal $\mathfrak{m} = (x, 3)$.

Outline

The Framework

We first recall Proposition 1 of [Fak01]. We will use the notations there.

Proposition

With the notations as before, we have $n \leq n_0 r p^t$
where n_0 *is the primitive period of the reduction* $\tilde{P}$ *of* P ,
r *is the order of the induced map on the cotangent space* $\mathfrak{m}/\mathfrak{m}^2$*,*
and t *depends only on* K *and the prime.*

Clearly $n_0 \leq |\tilde{\mathcal{X}}(k)|$.

Notations

Recall that r is the order of the induced map on $\mathfrak{m}/\mathfrak{m}^2$.
First we bound r.
We can show that $\dim_k(\mathfrak{m}/\mathfrak{m}^2) \leq d'$.
By a result of Darafsheh, we have $r \leq p^{d'} - 1$.

The Setup

Replacing f by an iterate we may assume that the reduction $\tilde{P}$ of P is fixed under f.
Let $\mathrm{Spec}(A)$ be the reduced subscheme of $\mathcal{X}$ determined by the orbit of P.
Then f induces an R-morphism σ of A.
Also let $\mathfrak{m}$ be the maximal ideal of A.
Suppose the induced map $\tilde{\sigma}$ on $\mathfrak{m}/\mathfrak{m}^2$ is the identity.

The Setup

As in [Fak01], we look at the induced map $\sigma : A \to A$.
Write $\sigma = \mathrm{id} + h$. Then $h(\mathfrak{m}) \subseteq \mathfrak{m}^2$.
Let $\nu : A \setminus \{0\} \to \mathbb{Z}$ be defined as follows:
for $0 \neq a \in A$, let $\nu(a)$ be the largest integer ℓ such that $a \in \mathfrak{m}^\ell$.

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for $0 \neq a \in A$, let $\nu(a)$ be the largest integer ℓ such that $a \in \mathfrak{m}^\ell$.
Since $h(\mathfrak{m}) \subseteq \mathfrak{m}^2$,
for all $a \in \mathfrak{m}$ either $h(a) = 0$ or $\nu(h^j(a)) > \nu(a)(j > 0)$.
We will show that the order of σ is a power of p.

Suppose a is of period s and we have

$$a = \sigma^s(a) = (\mathrm{id} + h)^s(a)$$

Poonen's Method

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$$= a + \binom{s}{1}h(a) + \binom{s}{2}h^2(a) + \cdots + \binom{s}{s-1}h^{s-1}(a) + h^s(a),$$
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Recall that our goal is to show that s is a power of p.
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Recall that our goal is to show that s is a power of p.
It suffices to show that either $s = 1$ or $p|s$.
Suppose $s \neq 1$ and p does not divide s.
Since $\nu(h^N(a)) > \nu(a)$ for $a \in \mathfrak{m}, N \geq 1$,
we must have $0 \notin \mathfrak{m}^{\nu(h(a))+1}$. Contradiction!

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Thank you!