Totally T-adic functions of small height

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Joint Mathematics Meetings
Arithmetic Dynamics Special Session

January 17, 2020

Center for Computing Sciences
17100 Science Drive • Bowie, Maryland 20715

Collaborators

This talk is joint work with Clayton Petsche at Oregon State University.

Totally Real Algebraic Numbers

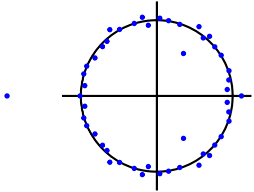
Def. $\alpha \in \bar{\mathbb{Q}}$ is totally real if its image under every embedding in $\mathbb{C}$ lies in $\mathbb{R}$.

e.g., $a + b\sqrt{2}$ for $a, b \in \mathbb{Q}$

Theorem (Schinzel)

If $\alpha \neq 0, \pm 1$ is totally real, then $h(\alpha) \geq \frac{1}{2} \log\left(\frac{1+\sqrt{5}}{2}\right)$, and this bound is sharp.

Why is it true? Bilu's equidistribution theorem says $h(\alpha_n) \to 0$ implies Galois orbits of α_n's equidistribute around unit circle in $\mathbb{C}$. But totally real numbers are stuck in the real axis.



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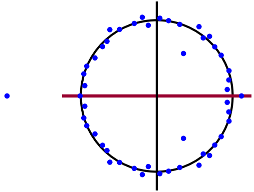

Totally p-adic Algebraic Numbers

Def. $\alpha \in \bar{\mathbb{Q}}$ is totally p-adic if its image under every embedding in $\mathbb{C}_p$ lies in $\mathbb{Q}_p$.

e.g., $a + b\sqrt{2}$ for $a, b \in \mathbb{Q}$, provided p odd and $\left(\frac{2}{p}\right) = 1$

Theorem (Bombieri/Zannier/Fili/Petsche/Pottmeyer)

If $\alpha \notin \{0\} \cup \mu_{p-1}$ is totally p-adic, then $h(\alpha) \geq \frac{\log(p/2)}{p+1}$.

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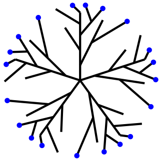

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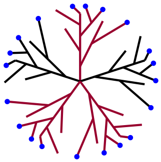

A Dramatic Reenactment

Function Field Heights

Let $K/\mathbb{F}_q(T)$ be a finite separable extension.

- K is the function field of a smooth proper curve C over $\mathbb{F}_q$.
- places of K correspond to closed points of C, denoted $|C|$
- For $P \in |\mathbb{P}^1_{\mathbb{F}_q}|$, set $\mathbb{C}_P = \widehat{\mathbb{F}_q(T)}^{\mathrm{sep}}$, completed with respect to ord_P.
  Height of $\alpha \in K$ is a sum of local contributions:

$$h(\alpha) = \frac{1}{[K : \mathbb{F}_q(T)]} \sum_{P \in |\mathbb{P}^1_{\mathbb{F}_q}|} \sum_{\sigma :\, K \hookrightarrow \mathbb{C}_P} \max\{-\mathrm{ord}_P(\sigma(\alpha)), 0\}.$$

- $h(\alpha) = 0$ if and only if $\alpha \in \mathbb{P}^1(\bar{\mathbb{F}}_q)$
- $\alpha \in K$ has a minimal polynomial $f \in \mathbb{F}_q[T][x]$. Then $h(\alpha) = \dfrac{\deg_T(f)}{\deg_x(f)}$.

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Totally T-adic Functions

Def. Set $\mathbb{C}_T = \widehat{\mathbb{F}_q(T)^{\mathrm{sep}}}$, completed with respect to ord_T.
Say $\alpha \in \mathbb{F}_q(T)^{\mathrm{sep}}$ is totally T-adic if

- its image under every embedding in $\mathbb{C}_T$ lies in $\mathbb{F}_q((T))$, or equivalently,
- T splits completely in the function field $\mathbb{F}_q(T, \alpha)$.

e.g., Roots of $Tx^{q+1} + x^q - x - cT$, where q is odd and $c \neq \square$ in $\mathbb{F}_q$.

Theorem (XF/Petsche)

If $\alpha \notin \mathbb{F}_q$ is totally T-adic, then $h(\alpha) \geq \frac{1}{q+1}$. This bound is sharp.

Proof Haiku. Rework Pottmeyer.
Archimedean places?
Not in function fields!

Don't like poetry? A geometric proof for pros(e)

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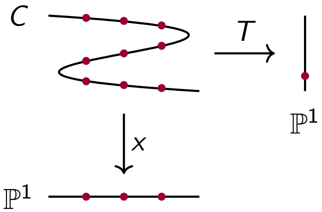
Proof. $f(T, x) =$ min. poly. for α.

Set $d = \deg_x(f)$ and $n = \deg_T(f)$.

$$\#C(\mathbb{F}_q)$$

$C_{/\mathbb{F}_q} =$ smooth proper curve birational to $\{f = 0\}$.



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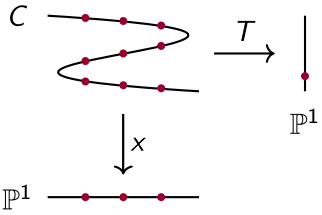

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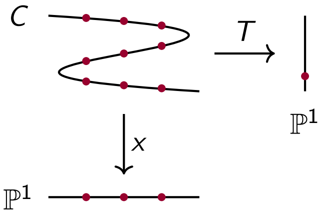

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$$\implies h(\alpha) = \frac{n}{d} \geq \frac{1}{q+1}. \qquad \text{Done!}$$

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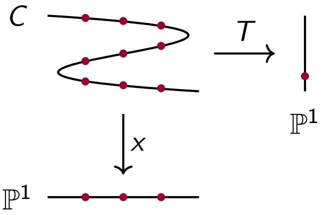

Geometry of Minimum Height Elements

Theorem. (XF/Petsche)

Suppose C is a smooth proper curve over $\mathbb{F}_q$, $n \geq 1$ is an integer, and $T, x \in \kappa(C)$ are separable functions such that:

- $\#C(\mathbb{F}_q) = n(q+1)$;
- T *has degree* $n(q+1)$ *and* x *has degree* n;
- $\kappa(C) = \mathbb{F}_q(T, x)$;
- T *vanishes at all points of* $C(\mathbb{F}_q)$.

Then x generates a totally T-adic extension of $\mathbb{F}_q(T)$, and $h(x) = \frac{1}{q+1}$.

Fun fact. Suppose $n \mid (q-1)$, and consider the cyclic n-cover of $\mathbb{P}^1$ given by

$$C_{/\mathbb{F}_q} : y^n = x^n(x^q - x)^n + 1.$$

For $T = \frac{x^q - x}{y}$, x is a totally T-adic function of minimum height.

Castelnuovo Hasse-Weil with Curves

Theorem. (XF/Petsche)

Let α be totally T-adic with minimum height $1/(q+1)$ and associated curve C. Then the gonality n and the genus $g(C)$ satisfy

$$\frac{(n-1)(q+1)}{2\sqrt{q}} \leq g(C) \leq \frac{1}{2}(q+1)(n-1)^2 + \frac{1}{2}(q-1)(n-1).$$

Lower bound. Hasse-Weil. $n(q+1) = \#C(\mathbb{F}_q) \leq q + 1 + 2g\sqrt{q}$

Upper bound. Essentially Castelnuovo's estimate. Functions T, x give a morphism $C \to \mathbb{P}^1 \times \mathbb{P}^1$. Use adjunction after controlling singularities.

Properties of Minimum Height Elements

Pottmeyer's argument plus invariance properties of totally T-adic numbers under the action of $\mathrm{PGL}_2(\mathbb{F}_q)$ show that the conjugates of a minimum-height α are "well-distributed". Over $\mathbb{F}_q[\![T]\!]$, the minimal polynomial of α factors as

$$f = \prod_{i=1}^{n}(Tx - a_i) \prod_{u\in\mathbb{F}_q} \prod_{i=1}^{n}(x - u - Tb_{u,i})$$

for some units $a_i, b_{u,i} \in \mathbb{F}_q[\![T]\!]^\times$.

Upshot. Working modulo T^{n+1} gives a search space for finding examples.

- Construct examples for $n = 3$ and $q = 3$:

$$T^3x^{12} + 2T^2x^{11} + \left(2T^3 + 2T\right)x^{10} + \left(T^2 + 1\right)x^9 + \left(T^2 + T\right)x^8 + \left(T^3 + 2T^2\right)x^7$$
$$+ \left(2T^3 + 2T\right)x^6 + 2T^3x^5 + \left(2T^3 + 2\right)x^3 + \left(2T^2 + T\right)x^2 + \left(T^3 + T^2\right)x + 2T^3.$$

- No example exists for $q = 2$ and $n = 2, 3, 4$!

What about elements of small height?

Write $\mathcal{T}_q$ for set of totally T-adic separable functions. Do we expect that $\liminf_{\alpha \in \mathcal{T}_q} h(\alpha) > \dfrac{1}{q+1}$? That's what happens for totally real numbers. Not sure about totally p-adic numbers.

Theorem. (XF/Petsche)

Set $\phi(x) = \frac{x^q - x}{T}$. Let $\alpha_j \in \mathbb{F}_q(T)^{\mathrm{sep}}$ satisfy $\phi^j(\alpha_j) = 1$. Then α_j is totally T-adic, and

$$\lim_{j \to \infty} h(\alpha_j) = \frac{1}{q-1} = \frac{1}{q+1} + \frac{2}{q^2-1}.$$

In particular,

$$\frac{1}{q+1} \leq \liminf_{\alpha \in \mathcal{T}_q} h(\alpha) \leq \frac{1}{q-1}.$$

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Ongoing Work

1. Fix q. Do there exist infinitely many totally T-adic functions of minimum height $1/(q+1)$? For any fixed gonality n, the answer is no. (Follows from algorithmic description.) A heuristic suggests answer is no for $q = 2$ and yes for $q > 2$!

2. What is the true value of the limit infimum? More generally, what are the accumulation points (in $\mathbb{R}$) of heights of totally T-adic functions? Can we get to these points with dynamical constructions?

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Thank you!