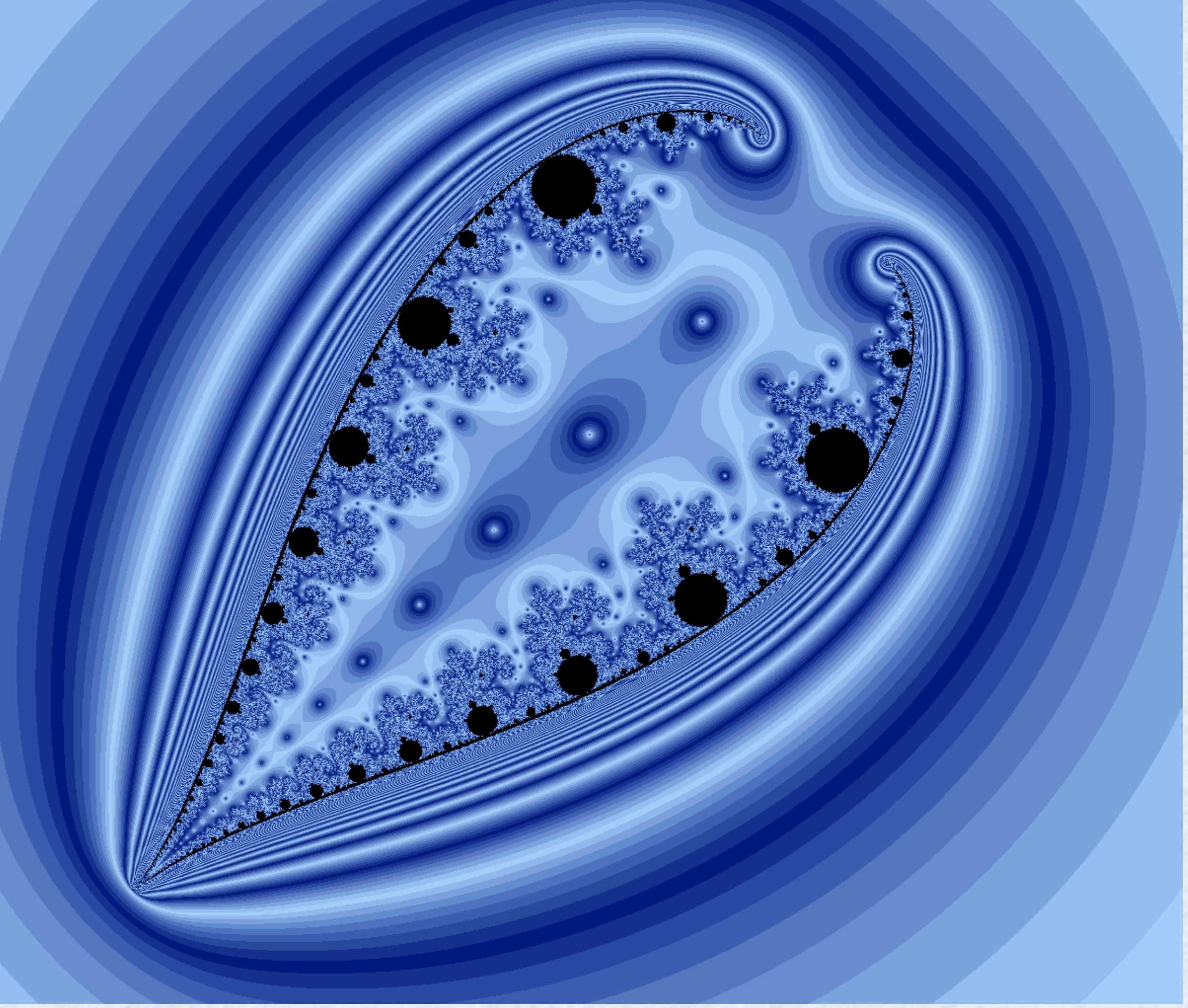

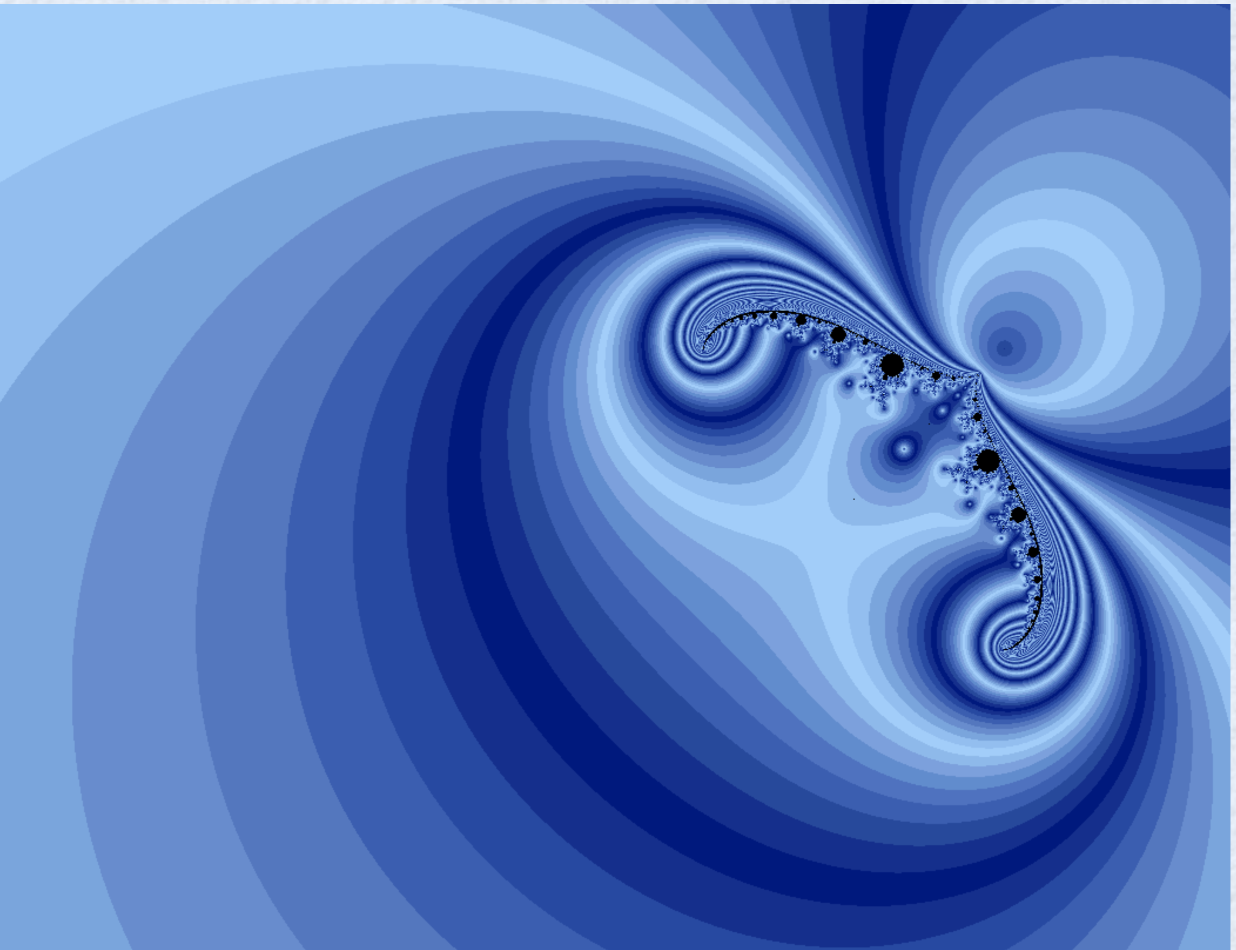

Multiplier spectra of PCF rational maps

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Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree $d \geqslant 2$.

The *multiplier* of the periodic cycle

$$x_0 \longrightarrow x_1 \longrightarrow \cdots \longrightarrow x_{n-1} \longrightarrow x_0$$

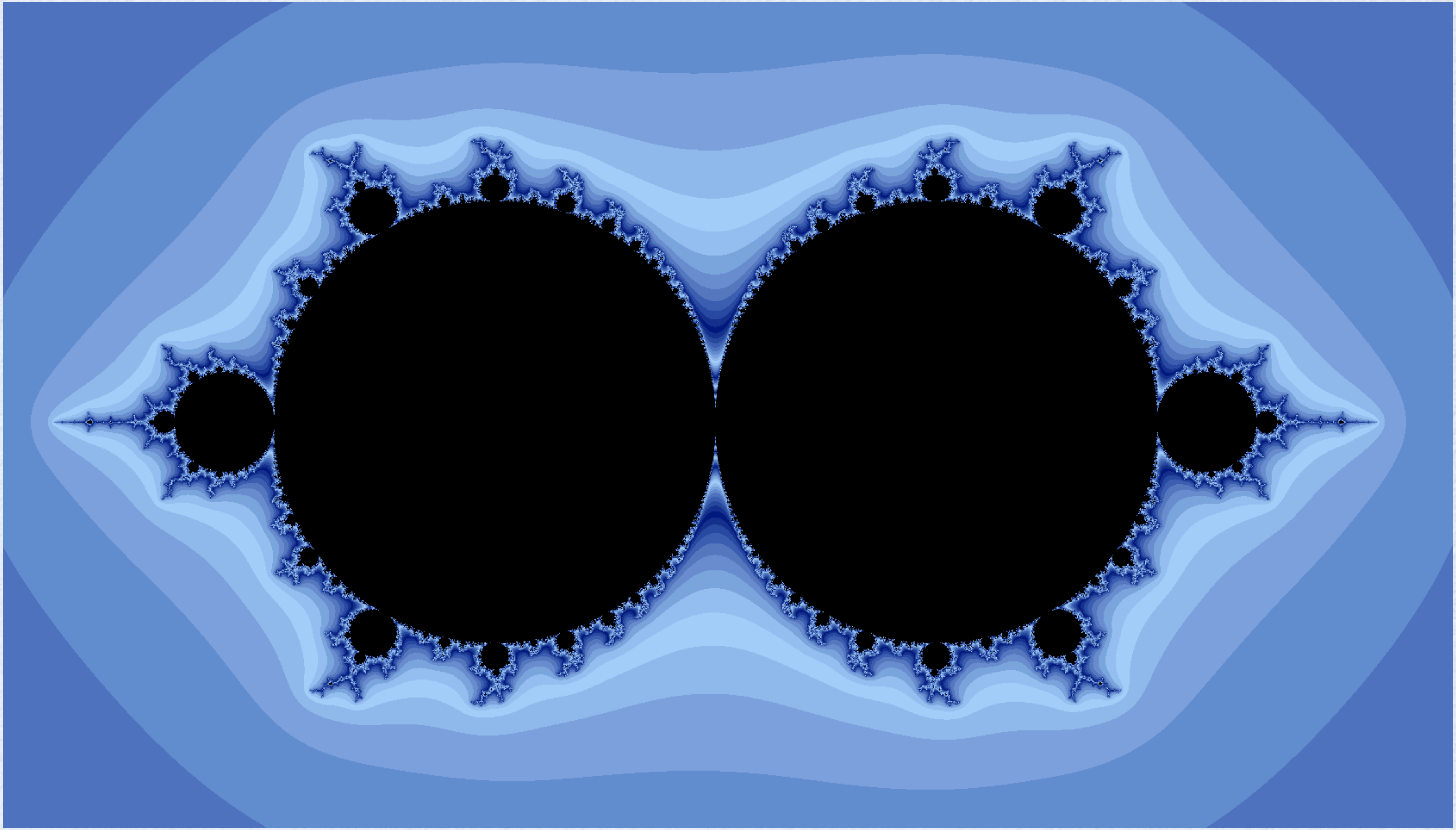
period n

is the complex number

$$\lambda = (f^n)'(x_i).$$

The *multiplier spectrum* of f is the subset of $\mathbb{C}$ consisting of the multipliers of all periodic cycles of f.

How do we understand Λ_f?



$f_a : z \mapsto az(z-1)$

$f_a(0) = 0$, and $f'_a(0) = a$.

Postcritically finite rational maps

Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree $d \geqslant 2$.

f is PCF provided that every critical point eventually maps into a periodic cycle

PCF rational maps are entitled to only repelling periodic cycles or superattracting cycles

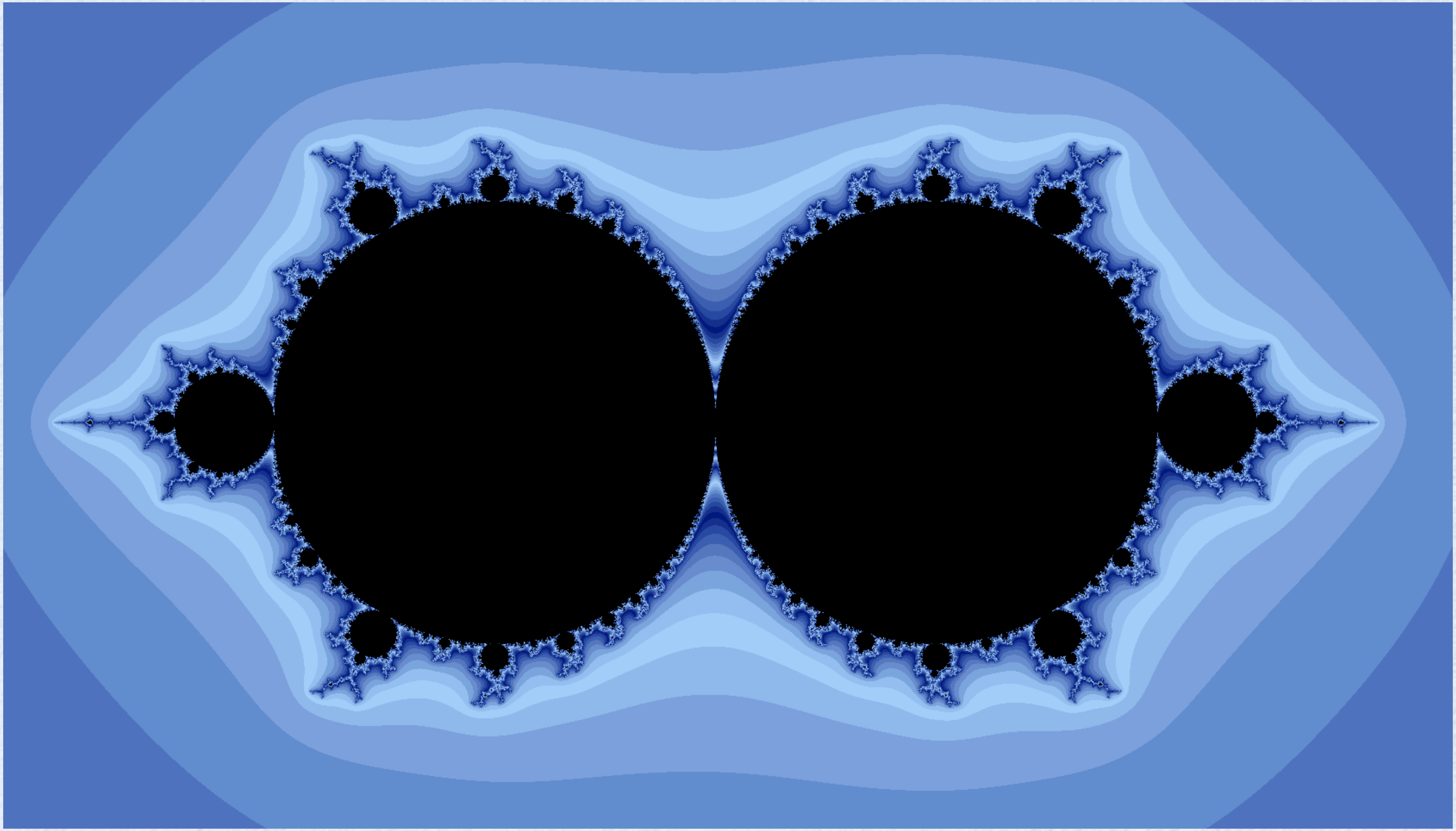
PCF rational maps have algebraic coefficients (Thurston)

$\Longrightarrow$ if λ belongs to the multiplier spectrum of a PCF rational map, then

$$\lambda \in \overline{\mathbb{Q}}, \quad \text{and} \quad \lambda = 0 \quad \text{or} \quad |\lambda| > 1$$

Galois conjugates

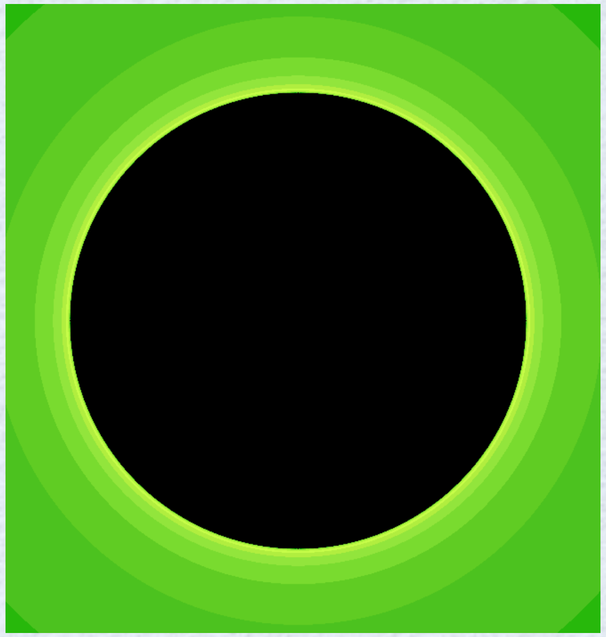
$$z \mapsto az(1-z)$$



PCF locus?

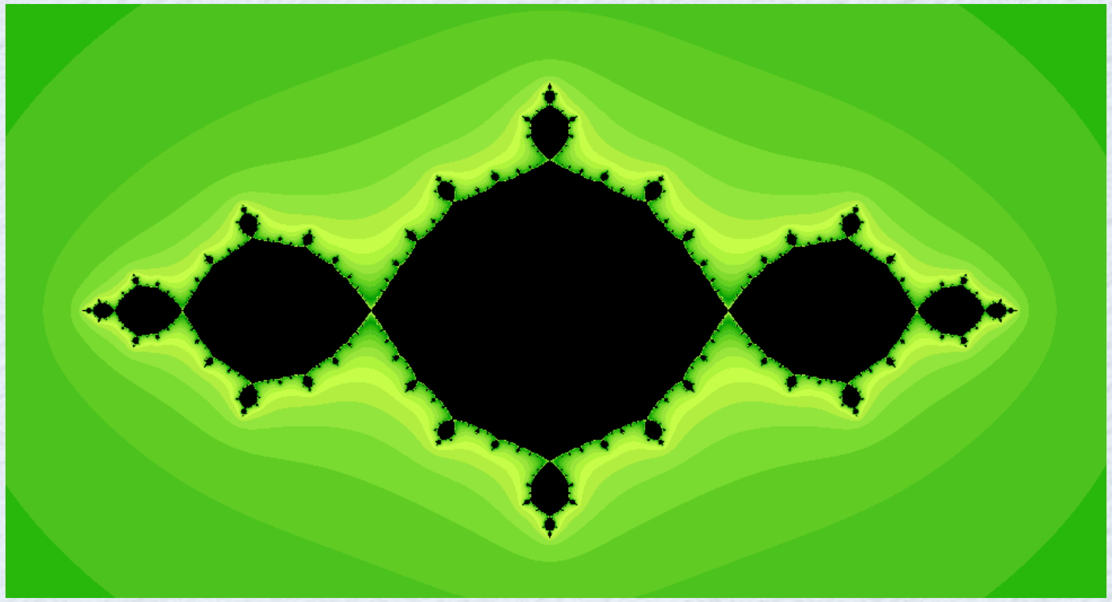
Exercise:

What can we say about the multiplier spectrum of $z \mapsto z^2$?



Question:

What can we say about the multiplier spectrum of $z \mapsto z^2 - 1$?



This is too hard.
Ask a different question.

Consider PCF rational maps of degree 2.
Can we say anything about the multiplier spectra?

a quadratic rational maps has 3 fixed points.

Lemma 3.1. *These three multipliers determine f up to holomorphic conjugacy, and are subject only to the restriction*

$$\mu_1\mu_2\mu_3 - (\mu_1 + \mu_2 + \mu_3) + 2 = 0, \qquad (3\text{–}1)$$

or, equivalently,

$$\sigma_3 = \sigma_1 - 2. \qquad (3\text{–}2)$$

Hence the moduli space $\mathcal{M}_2$ is canonically isomorphic to $\mathbf{C}^2$, with coordinates σ_1 and σ_2.

a quadratic rational map has a **unique** periodic cycle of period 2 (allowing degenerations).

$$1 \rightleftarrows c \qquad \infty \xrightarrow{2} v \longrightarrow \cdots$$

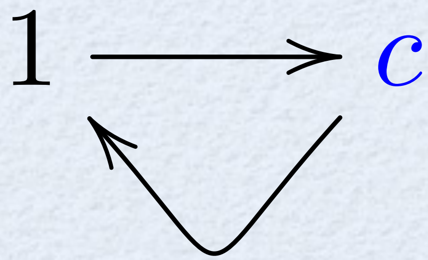
$$0 \xrightarrow{2} \cdots$$

$$f_{c,v} : z \mapsto \frac{(v-1)c^2 + (v-1)c + (1-z^2)v}{(v-1)c - z^2 + v}$$

The multiplier is:

$$\mu = \frac{4c}{(c+1)^2} \quad !!$$

Which values arise as this multiplier for PCF maps?



$$f_{c,v} : z \mapsto \frac{(v-1)c^2 + (v-1)c + (1-z^2)v}{(v-1)c - z^2 + v}$$

The multiplier is:

$$\mu = \frac{4c}{(c+1)^2}$$

Which values arise as this multiplier for PCF maps?

$\mathrm{Per}_2(\lambda) \subseteq \mathrm{M}_2$ is the set of all f that have a 2-cycle of multiplier λ. For which $\lambda \in \mathbb{C}$, is

$$\mathrm{Per}_2(\lambda) \cap \mathrm{PCF} \neq \emptyset?$$

$$\mathrm{Per}_2(\lambda) \cap \mathrm{PCF} = \emptyset?$$

$$\mathrm{Per}_2(\lambda) \cap \mathrm{PCF} \text{ finite}?$$

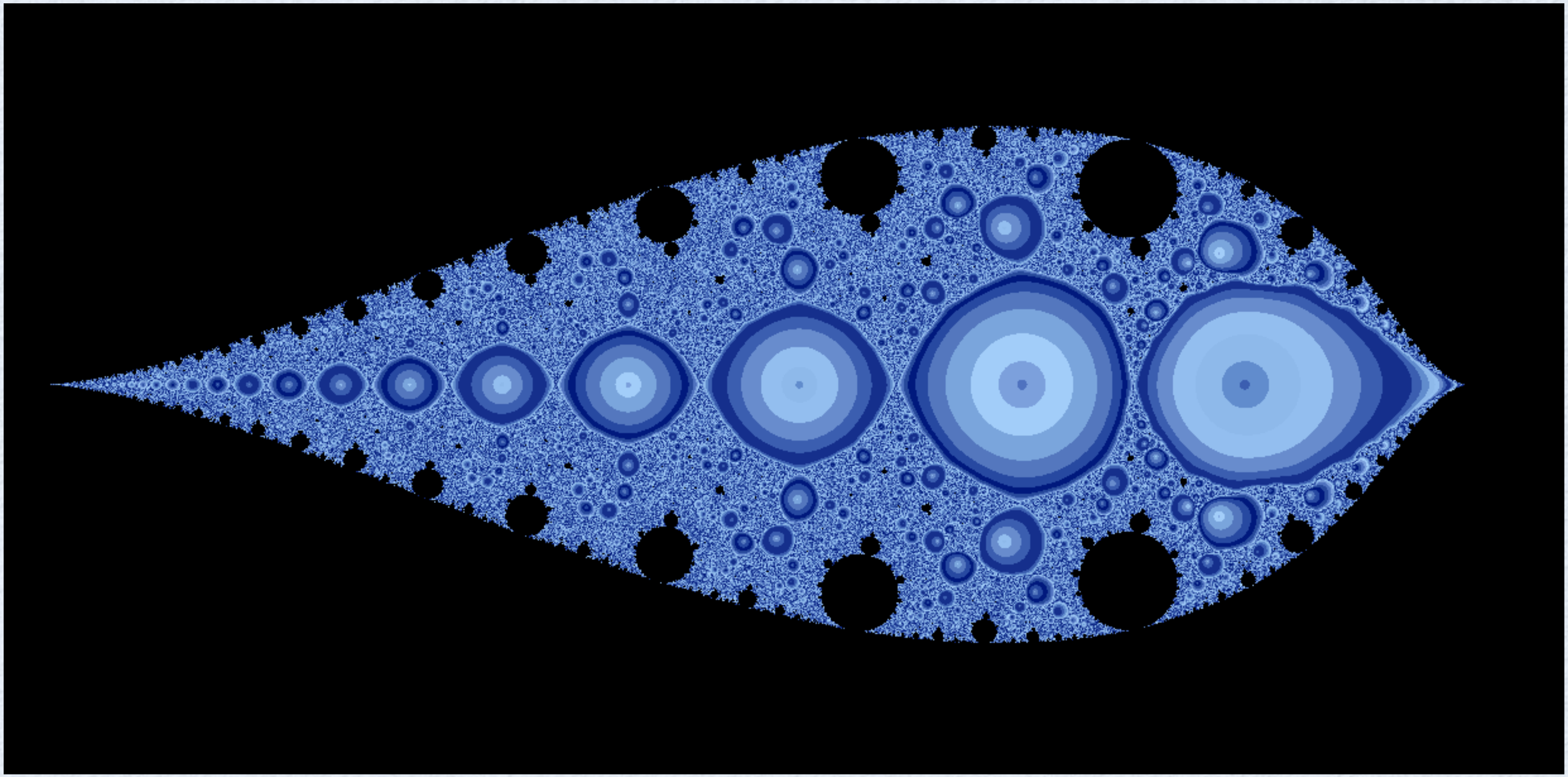
$$\mathrm{Per}_2(\lambda) \cap \mathrm{PCF} \text{ infinite}? \text{ (Baker-DeMarco)}$$

$$\mu = \frac{4c}{(c+1)^2}$$

$$1 \longrightarrow c \xrightarrow{2} 1$$

$$\infty \xrightarrow{2} v \longrightarrow \cdots$$

$$c = 0$$



$\mathrm{Per}_2(0)$

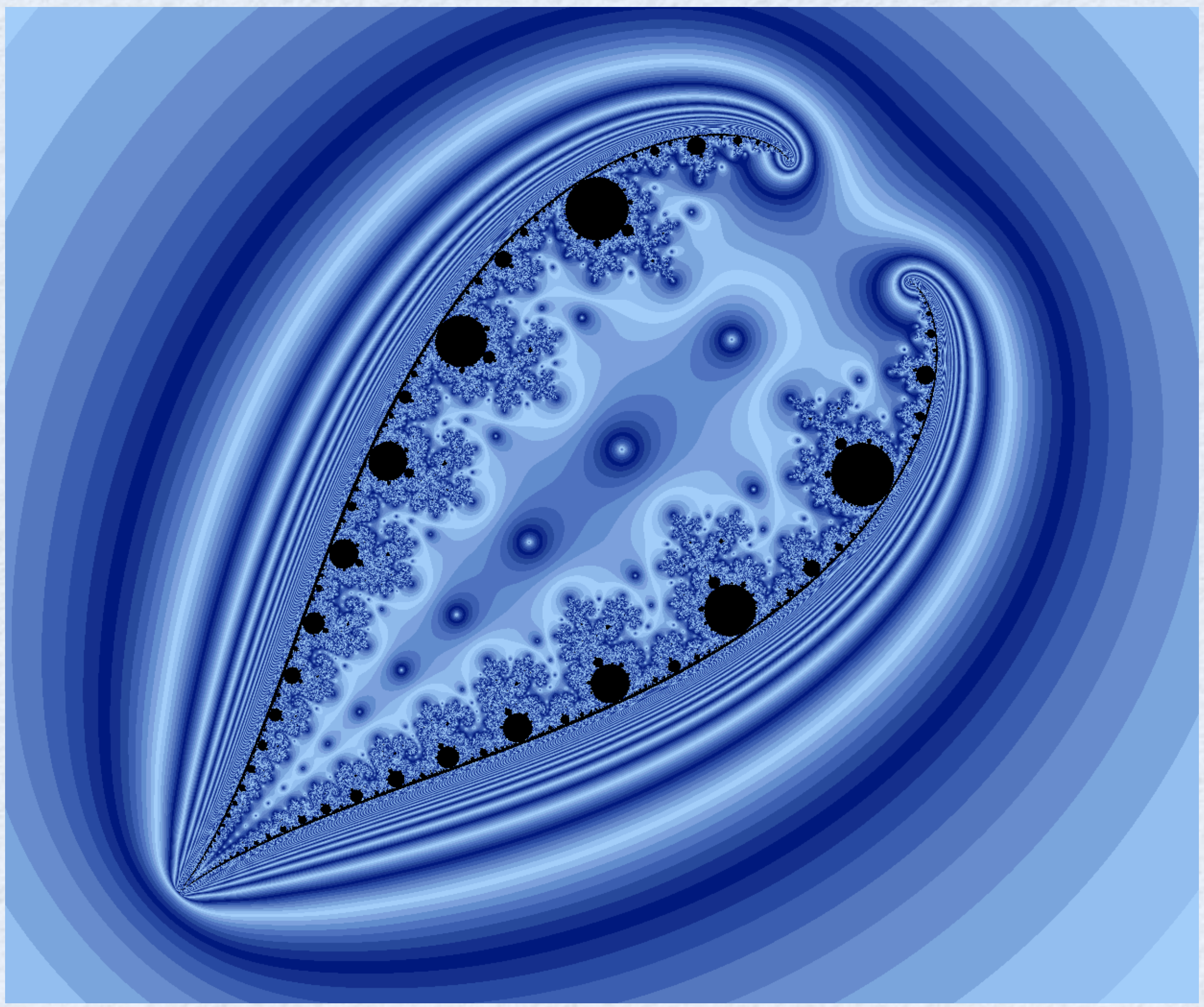
$$\mu = \frac{4c}{(c+1)^2}$$

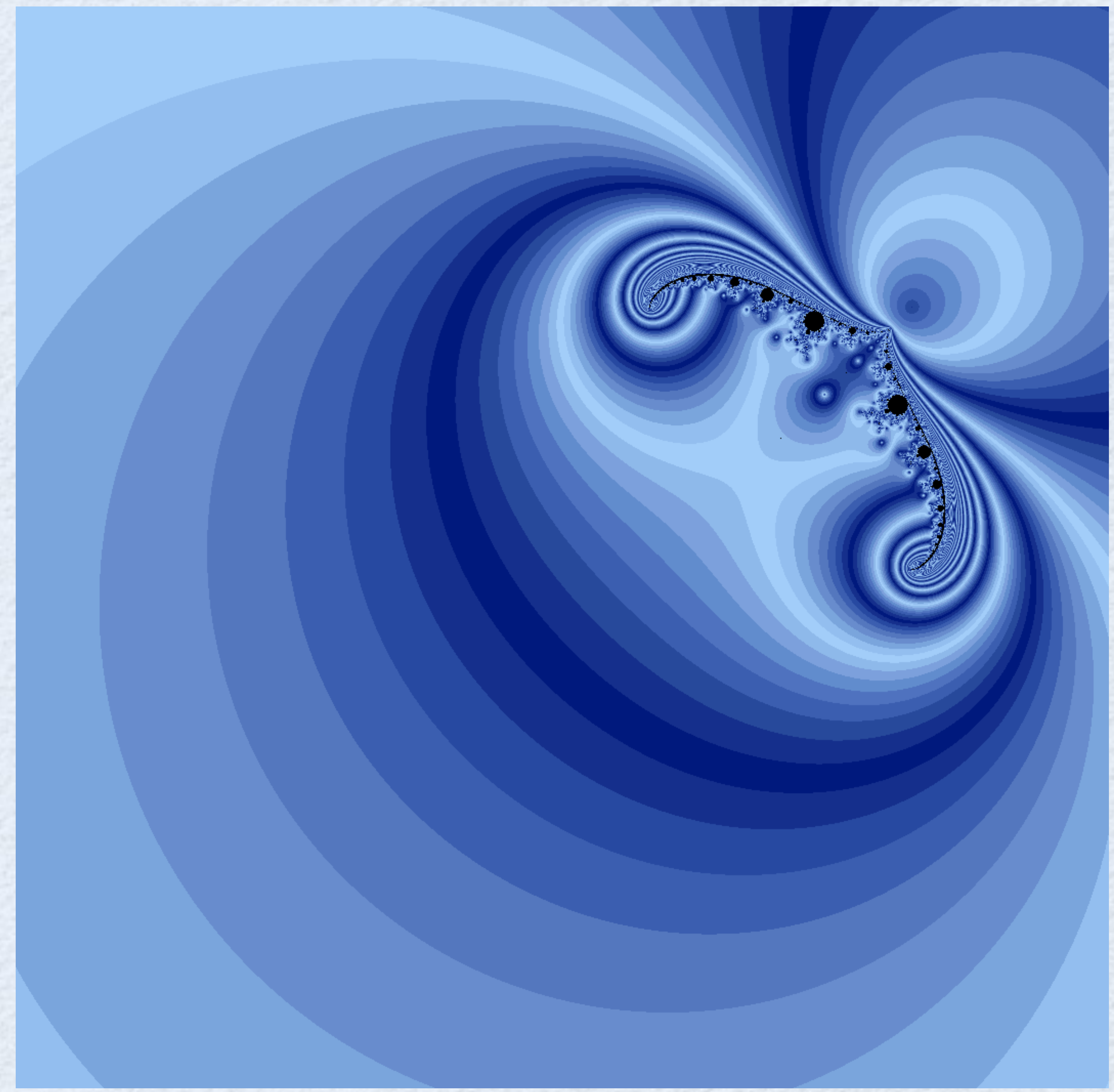
$$1 \longrightarrow c \longrightarrow 1$$

$$\infty \xrightarrow{2} v \longrightarrow \cdots$$

$$0 \xrightarrow{2} \cdots$$

$c = i$





$\mathrm{Per}_2(2)$

Are there any PCF maps in here?

Algebraic conspiracies

$$\mathrm{Per}_3(1) = \mathrm{Per}_1(\omega) \cup \mathrm{Per}_1(\bar{\omega}) \cup \mathrm{Per}_2(-3),$$

where ω is a primitive cube root of unity (compare Figures 7 and 10). The first two lines, with slope -1, correspond to maps for which one orbit of period 3 degenerates to a fixed point of multiplier ω or $\bar{\omega}$, while the third straight line, with equation $\sigma_2 = -2\sigma_1 - 3$, corresponds to maps for which the two period-three orbits coincide. For some reason, which I do not understand, this locus is precisely equal to the line $\mathrm{Per}_2(-3)$. This third

Consequently $\mathrm{Per}_2(-3) \cap \mathrm{PCF} = \emptyset$.

Algebraic conspiracies

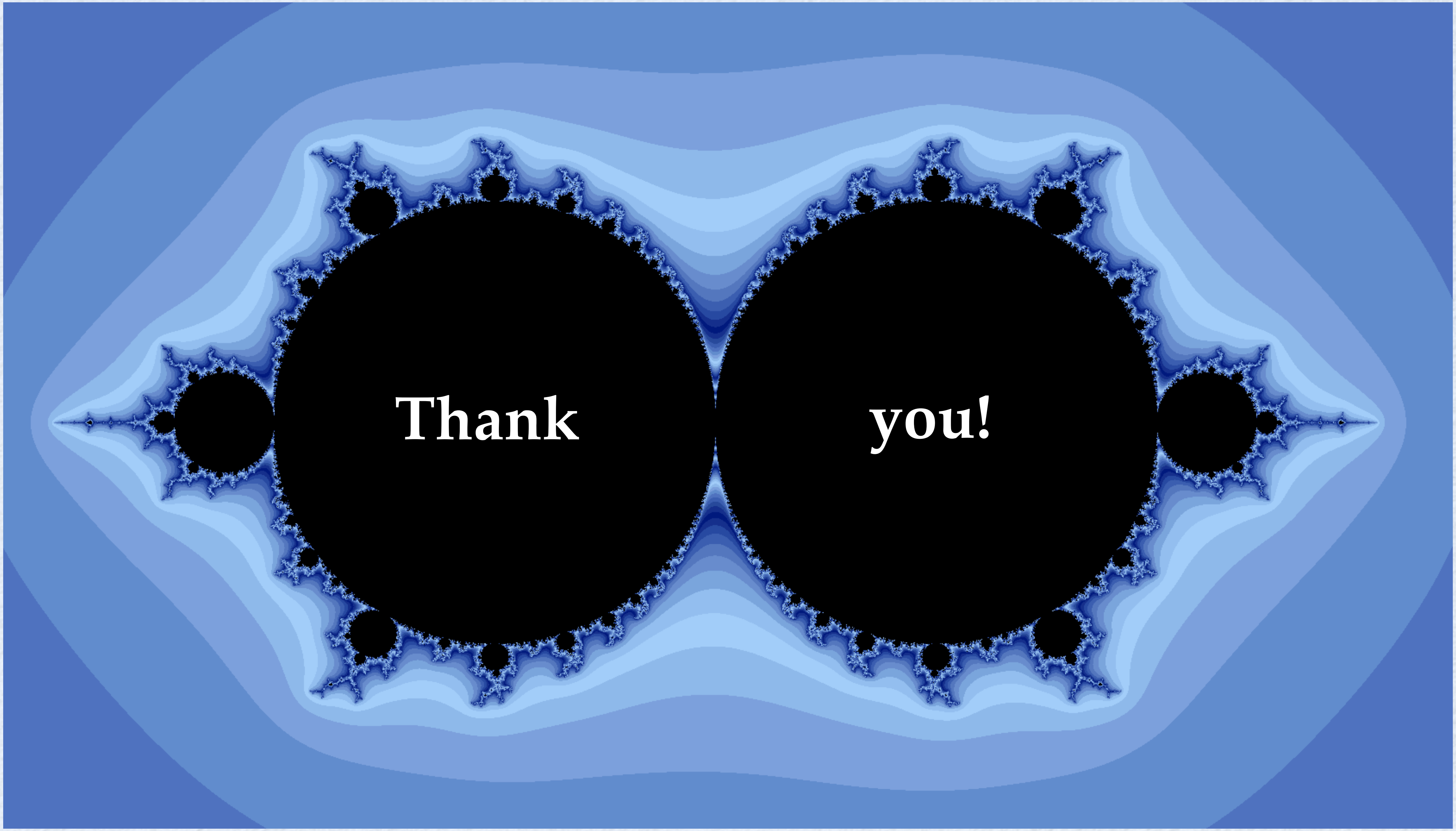
Do other curves $\mathrm{Per}_n(\lambda)$ and $\mathrm{Per}_m(\mu)$ ever coincide?

If $\mathrm{Per}_n(\lambda)$ and $\mathrm{Per}_m(\mu)$ coincide, then are λ and μ necessarily algebraic?

$$\mathbf{Per_3(1) = Per_1(\omega) \cup Per_1(\bar{\omega}) \cup Per_2(-3)},$$

What can we say about $\mathrm{Per}_k(1)$?

It should be "predictably" reducible.


Thank
you!