Portrait Spaces for Dynamical Semigroups and Unlikely Intersections

Preliminary Report

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Dynamical Semigroups

Goal: To understand the existence of finite orbits of dynamical semigroups.

- Let $f_1(x), f_2(x), \dots, f_m(x) \in \mathbb{C}(x)$ be rational functions,

  $\langle f_1, f_2, \dots, f_m \rangle := \{\text{Semigroup gen. by } f_i(x) \text{ under composition.}\}$

- If $p \in \mathbb{P}^1(\mathbb{C})$, we are interested in studying the **orbit** of p under $D = \langle f_1, \dots, f_m \rangle$,

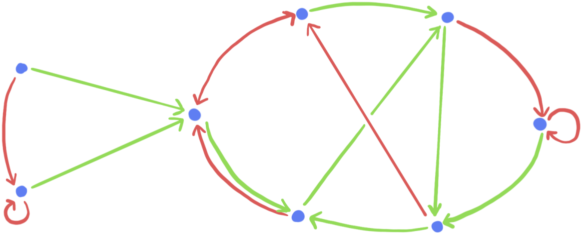
$$D(p) = \{w(p) : w \in D\}.$$

- If $D = \langle f(x) \rangle$, then

$$D(p) = \{p, f(p), f^2(p), \dots\}.$$

Finite Orbits

Portrait: A finite set P together with an action by a free semigroup $\langle F_1, F_2, \ldots, F_m \rangle$.



Realization: A finite set of points in a space X labelled by elements of P and endomorphisms $f_1, f_2, \ldots, f_m$ of X such that for all $p \in P$ and all f_j,

$$f_j(x_p) = x_{F_j(p)}.$$

Portrait Moduli Spaces

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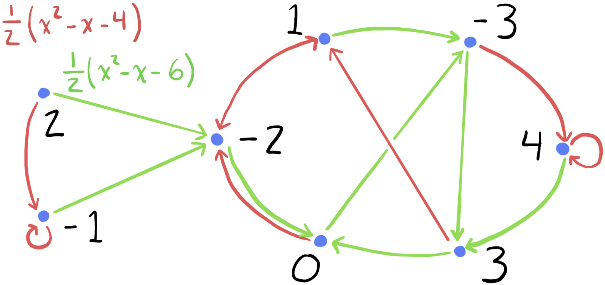

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Portrait Moduli Spaces

- If P is a portrait and X is a space, let $\mathcal{R}_P(X)$ denote the **realization space** of the portrait P in X.

  $$\mathcal{R}_P(X) = \{(f_1, \ldots, f_m; x_1, \ldots, x_n) : f_i \in \mathrm{End}(X),\ x_j \in X,\ \text{realize } P\}.$$

- The group $\mathrm{Aut}(X)$ acts naturally on $\mathcal{R}_P(X)$, let

  $$\mathcal{M}_P(X) := \mathcal{R}_P(X)/\mathrm{Aut}(X)$$

  denote the **portrait moduli space** of P in X.

Interesting Special Case

- Let $X = \mathbb{A}^1(\mathbb{C})$, so $\mathrm{End}(\mathbb{A}^1) = \mathbb{C}[x]$.
- Let $\mathcal{M}_P^d$ denote the subspace of $\mathcal{M}_P(\mathbb{A}^1(\mathbb{C}))$ where each $f_i(x)$ is a degree d polynomial.
- If P has $m = 2$ functions and $n = 2d$ points, then the expected dimension of $\mathcal{M}_P^d$ is 0.

Suppose P has m degree d polynomials and n points.

Expected Dimension of $\mathcal{M}_P^d$

$$= \#\text{coeffs.} + \#\text{pts.} - \#\text{arrows} - \dim \text{sym.}$$
$$= m(d+1) + n - mn - 2.$$

Two Quadratics, Four Points

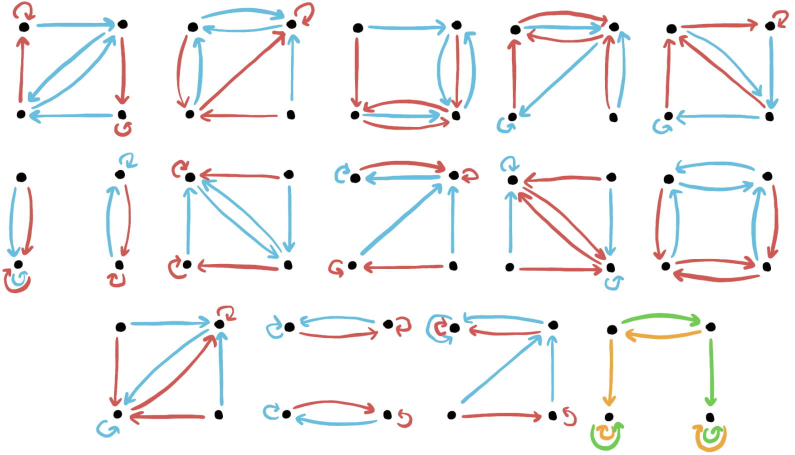
- Accounting for symmetry and degree constraints, there are 780 combinatorially distinct plausible portraits P.

Dim. of $\mathcal{M}_P^2$	Num. of Portraits
-1	206
0	560
1	14

- **Question:** Can we detect these unlikely intersections from the combinatorics of P?

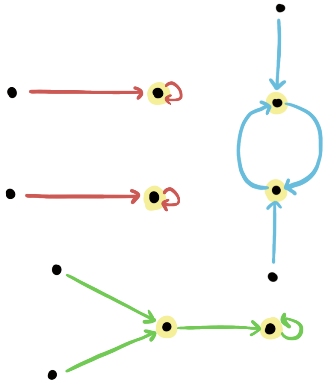
Two Quadratics, Four Points

Portraits P with $\mathcal{M}_P^2$ of dimension 1



Two Quadratics, Four Points

- 13 of the 14 portraits with $\dim \mathcal{M}_P^2 = 1$ were combinations of these 3 portraits:
- These are precisely the degree 2 portraits with **two image points**.



Two Quadratics, Four Points

Theorem (Two-Image Theorem)

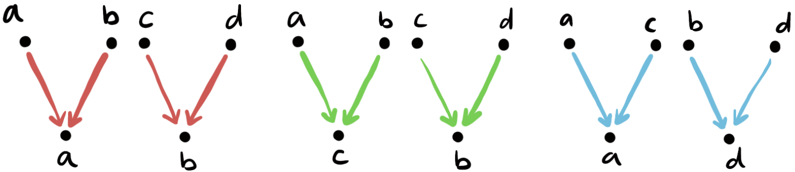
*Let $d \geq 1$ and suppose P is a plausible degree d portrait with $2d$ points and m functions $F_1, \ldots, F_m$ such that $|F_i(P)| = 2$ for each i. If the **fiber partitions***

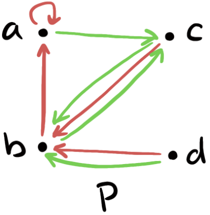
$$\Pi_i = \{F_i^{-1}(p) : p \in F_i(P)\}$$

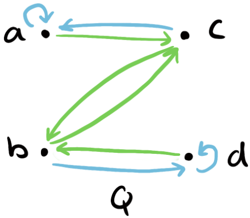
of P are all identical, then $\dim \mathcal{M}_P^d = d - 1$. Otherwise, $\mathcal{M}_P^d = \emptyset$.

- $\mathrm{Aut}(\mathbb{A}^1(\mathbb{C}))$ is sharply 2 transitive.
- Generalizes to rational functions with two replaced by three, and to other natural families.
- $d - 1$ is the maximal possible dimension.

Two Quadratics, Four Points





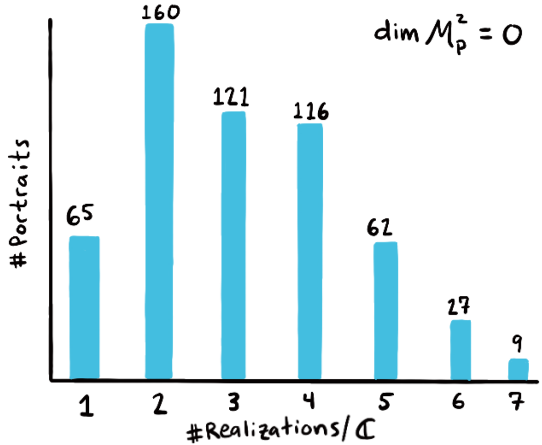


$\dim \mathcal{M}_P^2 = 1$

$\dim \mathcal{M}_Q^2 = -1$

Two Quadratics, Four Points

- If $\dim \mathcal{M}_P^2 = 0$, then $\mathcal{M}_P^2$ is a finite set of points.



$\mathbb{Z}$ Realizations	
Split Fld.	#Portraits
$\mathbb{Q}(\sqrt{-2})$	52
$\mathbb{Q}(\sqrt{2})$	49
$\mathbb{Q}(\sqrt{-7})$	33
$\mathbb{Q}(\sqrt{5})$	10
$\mathbb{Q}(\sqrt{-3})$	9
$\mathbb{Q}(\sqrt{6})$	4
$\mathbb{Q}(\sqrt{17})$	2
$\mathbb{Q}(\sqrt{-15})$	1

Two Cubics, Six Points

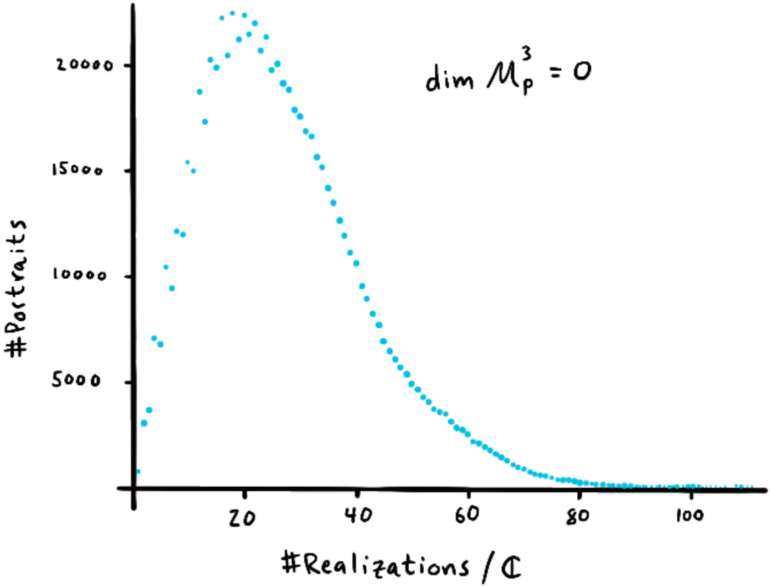
- 1350742 combinatorially distinct plausible portraits P.
- We have "computed" $\mathcal{M}_P^3$ for over 97% of portraits.

Dim. of $\mathcal{M}_P^3$	Num. of Portraits*
-1	42753
0	745608
1	909
2	11

- These portraits all accounted for by our Two-Image Theorem.

***Incomplete data!**

Two Cubics, Six Points



Thank you!

And thanks to ICERM for hosting us in the summer of 2019 and providing generous access to their computing resources!