A Néron symbol for arithmetic dynamics

Definitely not a talk about the critical height

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The canonical and critical heights on $\mathbb{P}^1$

Let h be the usual Weil height on $\mathbb{P}^1$.

Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ have degree $d \geq 2$.

Recall this guy:

$$\hat{h}_f(P) = \lim_{k\to\infty} \frac{h(f^k(P))}{d^k}$$

and his friend:

$$\hat{h}_{\mathrm{crit}}(f) = \sum_{P\in\mathbb{P}^1} (e_f(P) - 1)\hat{h}_f(P)$$

Critical height is a moduli height

Theorem (I. 2018, conjectured Silverman 2010)

If h is an ample Weil height on M_d, and $\mathsf{L}_d \subseteq \mathsf{M}_d$ is the locus of Lattès maps, then

$$h \asymp \hat{h}_{\mathrm{crit}} \qquad \text{on} \qquad \mathsf{M}_d \setminus \mathsf{L}_d.$$

See also Benedetto-I.-Jones-Levy 2014 which proves same when $\hat{h}_{\mathrm{crit}}(f) = 0$, I. 2012 which proves the complete statement for polynomials, and Tobin 1.5 hours ago for results about bicritical polynomials.

Heights for endomorphisms of $\mathbb{P}^N$

Now let $f : \mathbb{P}^N \to \mathbb{P}^N$ have degree $d \geq 2$.

There is a canonical height for points, but this critical locus is a divisor, defined by the usual $\det J_f = 0$.

There are several reasonable notions of height for divisors, including

$$h(F = 0) = \sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log \|F\|_v,$$

where $\|F\|_v$ is the v-adic sup norm on coefficients (in other words, just the height of D as a point in the appropriate dual space).

Zhang's height

Zhang constructed a canonical height

$$\hat{h}_f(D) = h(D) + O_f(\deg(D))$$

with

$$\hat{h}_f(f_* D) = \deg(f)^N \hat{h}_f(D)$$

(but he didn't normalize it like this)

This normalization gives $\hat{h}_f(D+E) = \hat{h}_f(D) + \hat{h}_f(E)$.

Hutz gives an alternate construction of Zhang's height from the functorial properties of Bost-Gillet-Soulé's heights from arithmetic intersection theory.

The limit definition

Post hoc, you can define the height as

$$\hat{h}_f(D) = \lim_{k \to \infty} \frac{h(f_*^k D)}{d^{Nk}}$$

but none of these sources make the rate of convergence (i.e., the error) explicit in terms of f.

One irksome thing is that the map on coefficients $D \mapsto f_* D$ is just a map of projective spaces, but the dimensions depend on the degree of the divisor.

The limit definition

With C_f denoting the critical divisor of f,

$$\hat{h}_{\mathrm{crit}}(f) := \hat{h}_f(C_f)$$

is a natural thing to look at on M_d^N, too.

It is well-defined (even modulo conjugacy), and well-behaved under iteration

$$\hat{h}_{\mathrm{crit}}(f^n) = n\hat{h}_{\mathrm{crit}}(f).$$

Is the critical height a moduli height? (again)

Conjecture

For any ample Weil height h on M_d^N, there exists a proper Zariski-closed $Z \subseteq \mathsf{M}_d^N$ such that

$$h \asymp \hat{h}_{\mathrm{crit}} \qquad \textit{on} \qquad \mathsf{M}_d^N \setminus Z.$$

+5 points if it's always the same Z
+25 points if Z has an explicit description.

But wait... can I even compute this thing?

Local estimates

Although one can proceed directly, the following widget is useful for proving local estimates...

Lemma (I., after Baker-Rumely)

Let $v \in M_K$. There exists a unique bilinear pairing

$$g_{f,v} : Z^1(\mathbb{P}^N_{\mathbb{C}_v}) \times Z_1(\mathbb{P}^N_{\mathbb{C}_v}) \to \mathbb{R}$$

with

$$g_{f,v}(f^* D, P) = g_{f,v}(D, f_* P),$$

$$g_{f,v}(f_* D, P) = g_{f,v}(D, f^* P),$$

$$g_{f,v}(D, P) = \langle D, P \rangle + O_{f,v}(1)$$

$$g_{f^\varphi,v}(D, P) = g_{f,v}(\varphi^* D, \varphi^* P)$$

for $\varphi \in \mathrm{Aut}(\mathbb{P}^N)$ and other properties which don't fit on the slide.

And yes, it comes in Berkovich flavours...

Local-to-global

Maybe most crucially, for our purposes,

$$\sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} g_{f,v}(D, P) = \deg(P)\hat{h}_f(D) + \deg(D)\hat{h}_f(P).$$

Explicit estimates on $\hat{h}_f(D) - h(D)$ come from summing the corresponding local estimates.

In particular, we get effectively computable estimates of the form

$$\hat{h}_f(D) - h(D) = O(\deg(D)h(f)),$$

where "effectively computable" means there's a Sage worksheet, but it's not pretty.

Variation of the “Néron pairing”

Lemma

Let f_t, D_t, P_t be families over B, with $\dim(B) = 1$. Then there exist Néron function λ_1, λ_2 on B so that

$$g_{f_t,v}(D_t, P_t) = \lambda_1(t) + o(\lambda_2(t)).$$

Compare to results of Favre for degeneration of the Lyapunov exponent.

Indeed, $g_{f,v}(C_f, P)$ is easily related to the Lyapunov exponent at complex places.

The critical height in one-parameter families

Globally, we have

Lemma (I., after Call-Silverman)

$$\hat{h}_{f_t}(D_t) = \left(\hat{h}_f(D) + o(1)\right) h(t),$$

where $o(1) \to 0$ as $h(t) \to \infty$, and $\hat{h}_f(D)$ is computed on the generic fibre, given $\dim(B) = 1$).

As a corollary,

$$\hat{h}_{\mathrm{crit}}(f_t) = \left(\hat{h}_{\mathrm{crit}}(f) + o(1)\right) h(t),$$

which is connected to the conjecture above... tenuously.

In particular, it **is** the conjecture for this family if $\hat{h}_{\mathrm{crit}}(f) \neq 0$.

Other cases... ?

Are there any other cases in which the conjecture can be proven? (by me?)

For $A \in \mathrm{GL_N}$ and $b \in \mathbb{A}^N$, let $f_{A,b} : \mathbb{P}^N \to \mathbb{P}^N$ be the morphism induced by

$$(X_0, \ldots, X_N) \mapsto (X_0^d, \ldots, X_N^d) \begin{pmatrix} A & 0 \\ b & 1 \end{pmatrix}.$$

So, like, $z \mapsto z^d + b$ when $N = 1$ and $A = 1$...

Theorem (I. WIP)

$$\hat{h}_{\mathrm{crit}}(f_{A,b}) = h(b) + O(h(A)).$$

Thank you.