The Galois–dynamics correspondence for unicritical polynomials

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Joint Mathematics Meetings 2020

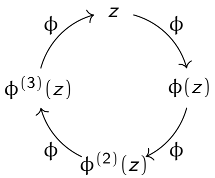
January 17, 2020

Overview

1. Introduction
   - Dynatomic modular curves
2. Galois–dynamics correspondence
   - Rationality of periodic points
   - Irreducibility criterion
   - Galois group criterion
   - Fixing the index in GDC
3. Known cases of GDC

Introduction

Goal: Characterize the periodic points of polynomials $\phi \in \mathbb{Q}[z]$.



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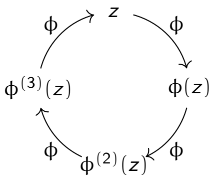

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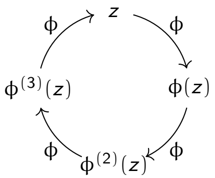

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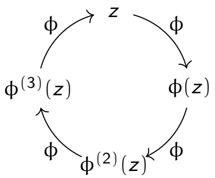

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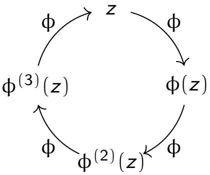

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Reduces to studying periodic points of $\phi_c := z^2 + c$ with $c \in \mathbb{Q}$:

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Extend the domain of z from $\mathbb{Q}$ to any number field K.

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Theorem (Douady–Hubbard ('85), Bousch ('92), Lau–Schleicher ('94), Xavier–Lei ('14), Morton ('96), Gao–Ou ('14))

$C_1(N)$ is smooth and geometrically irreducible (in characteristic 0).

Galois–dynamics correspondence

Definition

Fix $c \in \mathbb{Q}$. Let $K/\mathbb{Q}$ be a (nontrivial) finite Galois extension and let $N \geq 2$.
Suppose that for each N-periodic point $z \in K - \mathbb{Q}$ of ϕ_c,

$$\sigma z = \phi_c^{(i)}(z),$$

for some $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ and some positive integer $i < N$.
Then ϕ_c satisfies the *Galois–dynamics correspondence* (GDC) for K and N.

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For each cycle, the Galois–dynamics correspondence describes for some subgroup H of $\mathrm{Gal}(K/\mathbb{Q})$ and some n,

$$\text{Galois action} \longleftrightarrow \text{Dynamical acton}$$

$$H \longleftrightarrow \mathbb{Z}/n\mathbb{Z}$$

If the periodic point z is fixed, $H = \langle \sigma \rangle$ and $n = \frac{N}{\gcd(i,N)}$.

Rationality of periodic points

Theorem (Z.)

Suppose ϕ_c satisfies the Galois–dynamics correspondence for a quadratic number field K and N. Then:

- For any N-cycle $\{z_0, \ldots, z_{N-1}\}$ of ϕ_c in K, its trace $\sum_{i=1}^{N-1} \phi^{(i)}(z)$ is rational.
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Corollary for $d = 2$:
If the Galois–dynamics correspondence holds for a quadratic polynomial $\phi \in \mathbb{Q}[z]$, then ϕ has no *quadratic* periodic points of exact period 5 and exactly one (conditionally on standard conjectures) cycle of exact period 6.

Irreducibility criterion

Proposition (Vivaldi–Hatjispyros (1992))

Fix $c \in \mathbb{Q}$ and $d = 2$. Let $\phi = \phi_{2,c}$.
If $\Phi_N(z, c)$ is irreducible in $\mathbb{Q}[z]$ (not just $\mathbb{Q}[z, c]$), then ϕ_c satisfies the Galois–dynamics correspondence for all K.

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The proposition also holds for $d > 2$, but not much is known about irreducibility over $\mathbb{Q}[z]$.

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Proposition

Fix $c \in \mathbb{Q}$.
If the Galois group $G_{N,c}$ of $\Phi_N(z,c)$ over $\mathbb{Q}$ equals the Galois group G_N of $\Phi_N(z,t)$ over $\mathbb{Q}(t)$, then ϕ_c satisfies the Galois–dynamics correspondence for all K.

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In particular, more is known for small N:

- Σ_3 is infinite & explicitly described (Morton 1992)
- Σ_4 is infinite & explicitly described (Krumm 2018)
- $N \in \{5,6,7,9\} : \Sigma_N$ is finite (Krumm 2019)
- $N > 4$: Empirical evidence for finiteness

Fixing the index in GDC

Lemma

Let $K/\mathbb{Q}$ be a (nontrivial) finite Galois extension of degree D and z a periodic point of ϕ_c of exact period $N \geq 2$ in $K - \mathbb{Q}$. If for some $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$,

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Example

Assuming standard conjectures on $J(C_0(6))$, the only quadratic 6-cycle for any quadratic polynomial in $\mathbb{Q}[z]$ is:

$\phi_c = z^2 - \frac{71}{48}$ $\quad K = \mathbb{Q}(\sqrt{33})$ $\quad \sigma z = \overline{z}$ $\quad i = 3$

- $z_0 = -1 + \frac{\sqrt{33}}{12}$
- $z_1 = -\frac{1}{4} - \frac{\sqrt{33}}{6}$
- $z_2 = -\frac{1}{2} + \frac{\sqrt{33}}{12}$
- $z_3 = -1 - \frac{\sqrt{33}}{12}$
- $z_4 = -\frac{1}{4} + \frac{\sqrt{33}}{6}$
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Known cases of GDC for general d:

- $\Phi_N(z, c)$ is irreducible in $\mathbb{Q}[z]$ for fixed c: by an extension of Vivaldi–Hatjispyros
- $G_N \cong G_{N,c}$: exceptions Σ_N described for small N by Morton & Krumm