The Arakelov-Zhang pairing and Julia sets

Andrew Bridy (joint with Matt Larson)

Yale University

January 17, 2020

Notation

Let K be a number field.

Let h be the logarithmic Weil height on $\mathbb{P}^1(\overline{K})$.

For $\phi \in K(x)$, $\deg \phi \geq 2$, let $\hat{h}_\phi$ be the Call-Silverman canonical height:

$$\hat{h}_\phi(z) = \lim_{n\to\infty} \frac{h(\phi^n(z))}{(\deg \phi)^n}.$$

Let M_K be the set of places of K.

For $v \in M_K$, let P^1_v be the Berkovich projective line over $\mathbb{C}_v$.

Average heights of preimages

Our project started with the following experimental observation. Let $\phi(x) = x^2 + c$ for $c \in \mathbb{Z}$. if $|c| \geq 4$, then for any $\beta \in \overline{\mathbb{Q}}$, we computed

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{\phi^n(z)=\beta} h(z) = \hat{h}_\phi(0).$$

In general, our experiments suggested that the average height of preimages of β converges to the same number regardless of β, as long as β is non-exceptional (i.e., has infinite backward orbit). Moreover,

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{\phi^n(z)=\beta} h(z) \geq \hat{h}_\phi(0)$$

for every example we computed.

The Arakelov-Zhang pairing

Let $\psi, \phi \in K(x)$ for a number field K, with both $\deg \psi$ and $\deg \phi \geq 2$.

The Arakelov-Zhang pairing $\langle \psi, \phi \rangle$ is symmetric and non-negative. Its (rather technical) definition is in terms of local analysis on P^1_v at each $v \in M_K$. A simple way to characterize the pairing is:

Theorem (Petsche-Szpiro-Tucker)

Let (x_n) be any sequence of distinct points in $\mathbb{P}^1(\overline{K})$ such that $\hat{h}_\phi(x_n) \to 0$ as $n \to \infty$. Then $\hat{h}_\psi(x_n) \to \langle \psi, \phi \rangle$ as $n \to \infty$.

By this characterization, it appears somewhat miraculous that the pairing is not just well-defined, but symmetric!

The Arakelov-Zhang pairing

$\langle \psi, \phi \rangle$ may be interpreted as a “dynamical distance” between ψ and ϕ.

Theorem (Petsche-Szpiro-Tucker, Zhang, Chambert-Loir-Thuillier, Mimar, Baker-DeMarco...)

The following are equivalent:

1. $\langle \psi, \phi \rangle = 0$
2. $\hat{h}_\psi = \hat{h}_\phi$
3. $PrePer(\psi) = PrePer(\phi)$
4. $PrePer(\psi) \cap PrePer(\phi)$ *is infinite*
5. $\liminf_{x \in \mathbb{P}^1(\overline{K})} \hat{h}_\psi(x) + \hat{h}_\phi(x) = 0$

The pairing coincides with the square of a metric of mutual energy on adelic metrized line bundles (Fili) and has been used to study uniform Manin-Mumford problems (DeMarco-Krieger-Ye).

Average heights of preimages again

Using the transformation property $\hat{h}_\phi(\phi(x)) = (\deg\phi)\hat{h}_\phi(x)$, we can find a good example of a sequence of points (x_n) with $\hat{h}_\phi(x_n) \to 0$. Choose $\beta \in \mathbb{P}^1(K)$, and list the elements of

$$\phi^{-1}(\beta),\ \text{then } \phi^{-2}(\beta),\ \text{then } \phi^{-3}(\beta),\ \text{then } \phi^{-4}(\beta), \ldots$$

Choose $\psi(x) = x^2$, so that $\hat{h}_\psi = h$. Then, as long as β is non-exceptional, the PST characterization gives

$$\langle x^2, \phi \rangle = \lim_{n\to\infty} \frac{1}{(\deg\phi)^n} \sum_{\phi^n(z)=\beta} h(z),$$

where the sum is counted with multiplicity. If β is periodic for ϕ, we need to show that repeated roots of $\phi^n(z) = \beta$ don't contribute too much to the average – this can be done using equidistribution.

The canonical invariant measure

Let $v \in M_K$. If v is archimedean, let $\mu_{\phi,0}$ be the Haar measure on the unit circle in $\mathbb{P}^1(\mathbb{C})$. If v is non-archimedean, let $\mu_{\phi,0}$ be the point mass at the Gauss point on P^1_v.

The canonical ϕ-invariant probability measure $\mu_{\phi,v}$ is the weak limit of the sequence of measures defined by

$$\mu_{\phi,k+1} = \frac{1}{\deg \phi} \phi^* \mu_{\phi,k}.$$

The support of $\mu_{\phi,v}$ is precisely the Julia set $J(\phi)$ of $\phi : \mathsf{P}^1_v \to \mathsf{P}^1_v$. The multisets of nth preimages of any non-exceptional β equidistribute along $J(\phi)$ (Baker-Rumely, Chambert-Loir, Favre-Rivera-Letelier).

The main theorem

As $\hat{h}_{x^2} = h$, the pairing $\langle x^2, \phi \rangle$ may be interpreted as a measure of the dynamical complexity of ϕ. Our experimental observation leads to a way of proving a formula for $\langle x^2, \phi \rangle$.

Theorem (B.-Larson)

$$\langle x^2, \phi \rangle = h_\phi(0) - \sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \int_{|\alpha|<1} \log|\alpha| d\mu_{\phi,v}$$

We give a straightforward proof using equidistribution of average heights of preimages and basic algebraic number theory[1]. A shorter proof can be obtained by invoking more of the local analytic machinery.

[1] which doesn't work when 0 is in the Julia set of ϕ at some place

A corollary of the main theorem

Corollary

Let $\phi \in K(x)$. Then

$$\langle x^2, \phi \rangle \geq h_\phi(0)$$

with equality if and only the Julia set of $\phi : \mathsf{P}^1_v \to \mathsf{P}^1_v$ is disjoint from the open unit disk in P^1_v at every $v \in M_K$.

Note the disjointness hypothesis of (1) is satisfied at the non-archimedean place v if ϕ has potentially good reduction at v, i.e., good reduction after a change of variables.

An application of the corollary

Let $\phi(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in K[x]$, and assume $0 \in \mathrm{PrePer}(\phi)$.

Proposition

Suppose that for every non-archimedean place v, either

- *ϕ has potentially good reduction at v, or*
- *$v(a_0) \leq 0$ and $v(a_0) < v(a_i)$ for $1 \leq i \leq d-1$.*

Further suppose that the Julia set of ϕ at every archimedean place v does not intersect the open unit disc. Then $\phi(x) = x^d$.

The hypotheses imply $\langle x^2, \phi \rangle = 0$, thus $\hat{h}_\phi = \hat{h}_{x^2}$. Then we use a result of Kawaguchi-Silverman on polynomials with equal canonical heights.

This proposition gives us an easy proof of the following fact: if c is a preperiodic parameter in the Mandelbrot set, then $J(x^2 + c')$ must intersect the open unit disc for some conjugate c' of c.

Another corollary: the purely archimedean case

Corollary (2)

Let $\phi \in \mathbb{Z}[x]$ be monic. Then

$$\langle x^2, \phi \rangle = h_\phi(0) - \int_{|z|<1} \log |z| d\mu_\phi.$$

This explains our experimental observation about quadratic polynomials. Once $|c|$ is large enough, the complex Julia set of $x^2 + c$ is disjoint from the open unit disc.

The statement of the corollary is explicit enough to exactly compute $\langle x^2, \phi \rangle$ in some special cases. We also use it to improve bounds of Petsche-Szpiro-Tucker on $|h(x) - \hat{h}_\phi(x)|$ for some ϕ.

Chebyshev polynomials

For $n \geq 2$, let $T_n(x)$ be the degree n Chebyshev polynomial. The Julia set $J(T_n)$ is the closed interval $[-2, 2]$ in $\mathbb{R}$, and

$$d\mu_{T_n} = \frac{1}{2\pi} \frac{1}{\sqrt{1 - x^2/4}} \, dx$$

for the Lebesgue measure dx on $\mathbb{R}$. By Corollary 2,

$$\langle x^2, T_n \rangle = -\frac{1}{2\pi} \int_{-1}^{1} \frac{\log |x|}{\sqrt{1 - x^2/4}} dx = \frac{3\sqrt{3}}{4\pi} L(2, \chi) \approx 0.3231,$$

where L is the Dirichlet L-function associated to the nontrivial character χ mod 3.

The Arakelov-Zhang pairing and Julia sets

Andrew Bridy (joint with Matt Larson)

Yale University

January 17, 2020