Arboreal Cantor actions

Olga Lukina

University of Vienna

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Arboreal representations

Let K be a number field, f a polynomial of degree $d \geq 2$ and $\alpha \in K$.

Let K^{sep} be a separable closure of K. Under certain conditions, one can associate to K, f and α the representation

$$\rho_{f,\alpha} : \mathrm{Gal}(K^{sep}/K) \to Aut(T_d),$$

where T_d is a d-ary tree, and $Aut(T_d)$ is the automorphism group of T_d.

The group $G_\infty = \rho_{f,\alpha}(\mathrm{Gal}(K^{sep}/K))$ is a profinite (Cantor) group.

G_∞ acts on the path-space of T_d, which is a Cantor set.

Actions of countable and profinite groups on Cantor sets are also an object of interest in topological dynamics, geometric group theory, and other areas of mathematics.

Problem

Let G_∞ be a profinite group acting on a Cantor set X. Find invariants which measure the complexity of such an action.

In this talk, we introduce direct limit invariants of such actions, and use them to distinguish different types of actions. Based on

S. Hurder and O. Lukina, Limit group invariants for non-free Cantor actions, arXiv:1904.11072, submitted.

Open problem

Relate the invariants above to the number-theoretical properties of arboreal representations.

Let $d \geq 2$ be a positive integer.

The vertex set of a tree T_d is $V = \sqcup_{n \geq 0} V_n$, where V_n is a *level* of T_d.

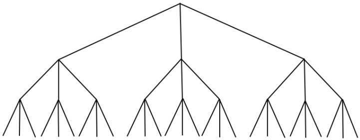
We have $|V_0| = 1$ and $|V_n| = d^n$, for $n \geq 1$, and each $v \in V_{n-1}$ is joined by edges to d vertices in V_n.

The space of paths ∂T_d consists of infinite sequences $\mathbf{v} = (v_0, v_1, v_2, \ldots)$, such that v_n and v_{n-1} are joined by an edge.

For paths (v_n) and (w_n), let $k = \max\{n \mid v_n = w_n\}$ and

$$D((v_n), (w_n)) = \frac{1}{2^k}.$$

Lemma: The space ∂T_d is a Cantor set, that is, compact totally disconnected metric space without isolated points.



Back to the polynomial $f(x)$ with coefficients in a field K:

Let $\alpha \in K$, and let $f^n(x) = f \circ f^{n-1}(x)$ be the n-th iteration.

Suppose $f^n(x) - \alpha$ is irreducible over K and has d^n distinct roots.

Adjoin these roots to K to obtain a finite extension $K(f^{-n}(\alpha))/K$.

Then the action of the Galois group

$$H_n = \mathrm{Gal}(K(f^{-n}(\alpha))/K)$$

permutes the roots of $f^n - \alpha$ and this action is transitive.

To obtain a tree T_d, let $V_n = \{ \text{ roots of } f^n(x) - \alpha\}$.

Then H_n acts on V_n transitively by permutations.

Join $\alpha \in V_n$ and $\beta \in V_{n-1}$ by an edge if and only if $f(\alpha) = \beta$.

There are homomorphisms $H_n \to H_{n-1}$, compatible with the tree structure, and the arboreal representations is the profinite group

$$G_\infty = \varprojlim \{H_n \to H_{n-1}\} \subset Aut(T_d).$$

Every automorphism of T_d induces a homeomorphism of the set ∂T_d of infinite paths in T_d, so we can view G_∞ as a subgroup of $Homeo(\partial T_d)$.

The space ∂T_d is a Cantor set with an ultrametric D, and G_∞ acts on ∂T_d by isometries.

So we pass from working with automorphisms of a tree T_d to working with isometries of its boundary (the space of paths) ∂T_d.

Elements with fixed points

Let ∂T_d be a path space of a d-ary tree, and let G_∞ be a profinite group acting transitively on ∂T_d.

To study actions, we look at elements which fix a given path $\mathbf{v} \in \partial T_d$.

These elements are of two types:

1. $g \in G_\infty$ such that $g \cdot \mathbf{v} = \mathbf{v}$ and $\mathbf{v}$ has an open neighborhood $U \subset \partial T_d$ such that for any $\mathbf{w} \in U$ we have $g \cdot \mathbf{w} = \mathbf{w}$.
2. $g \in G_\infty$ such that $g \cdot \mathbf{v} = \mathbf{v}$ and for any open neighborhood $U \subset \partial T_d$ of $\mathbf{v}$ the restriction $g|U$ is not the identity map.

We consider the behavior of such elements on the increasingly small neighborhoods of the point $\mathbf{x} \in \partial T_d$.

We see this as an analogue to studying infinitesimal dynamical properties of the system in the absence of derivatives.

Elements which fix a given path $\mathbf{v} = (v_0, v_1, v_2, \ldots) \in \partial T_d$ form the isotropy subgroup of the action

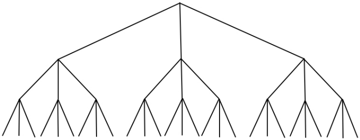
$$\mathcal{D}_{\mathbf{v}} = \{\widehat{h} \in G_\infty \mid \widehat{h} \cdot \mathbf{v} = \mathbf{v}\} \subset G_\infty,$$

called the *discriminant group* (not the same notion as the discriminant group of a polynomial!).

The discriminant group may be trivial, finite or a profinite (Cantor) group.

Then ∂T_d is a homogeneous space

$$\partial T_d \cong G_\infty / \mathcal{D}_{\mathbf{v}}.$$



For $v_n \in V_n$, the branch of the tree through v_n contains all paths in

$$U_{n,v_n} = \{(w_0, w_1, w_2, \ldots) \mid w_n = v_n\},$$

which is also a Cantor set.

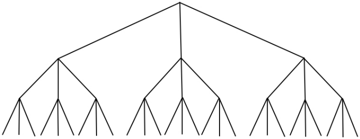
Elements which preserve U_{n,v_n} form a subgroup $\widehat{U}_{n,v_n} \subset G_\infty$ and

$$U_{n,v_n} \cong \widehat{U}_{n,v_n}/\mathcal{D}_{\mathbf{v}}.$$

The adjoint action of $\mathcal{D}_{\mathbf{v}}$ on $\widehat{U}_{n,v_n}$ is well-defined

$$Ad_n : \mathcal{D}_{\mathbf{v}} \times \widehat{U}_{n,v_n} \to \widehat{U}_{n,v_n} : (\widehat{h}, \widehat{g}) \mapsto \widehat{h}\widehat{g}\widehat{h}^{-1}.$$

Lemma: If $Ad_n(\widehat{h}) = id$, then $\widehat{h}$ fixes every point in U_{n,v_n}.



For $n \geq 0$, define the groups

$$K_n = \{ \widehat{h} \in \mathcal{D}_{\mathbf{v}} \mid \widehat{h}|U_{n,v_n} = id\} \qquad (\widehat{h} \text{ fixes every point in } U_{n,v_n})$$

$$Z_n = \{\widehat{h} \in \mathcal{D}_{\mathbf{v}} \mid Ad(\widehat{h})|\widehat{U}_{n,v_n} = id\} \qquad (Ad(\widehat{h}) \text{ fixes every point in } \widehat{U}_{n,v_n}).$$

Then $Z_n \subset K_n$, for all $n \geq 0$.

Form direct limit groups with respect to inclusions $\iota^n_{n+1} : K_n \to K_{n+1}$

$$\mathcal{K} = \sqcup_{n \geq 0} K_n / \{a \sim b \text{ iff } \iota^k_s(a) = \iota^m_s(b),\ s \geq k, m\},$$

$$\mathcal{Z} = \sqcup_{n \geq 0} Z_n / \{a \sim b \text{ iff } \iota^k_s(a) = \iota^m_s(b),\ s \geq k, m\}.$$

Theorem (Hurder and Lukina 2019)

The isomorphism classes of the direct limit groups $\mathcal{K}$ and $\mathcal{Z}$ are invariants of the conjugacy class of $(\partial T_d, G_\infty)$.

We say that the direct limit group $\mathcal{K}$ is *bounded*, if there exists $m \geq 0$ such that for all $n \geq m$ the inclusions $\iota : K_n \to K_{n+1}$ are isomorphisms.

The following classification was introduced in **Hurder and Lukina 2019**:

Stable and wild actions

A minimal equicontinuous group action $(\partial T_d, G_\infty)$ is *stable* if the group chain $\mathcal{K}$ is bounded, and the action is *wild* otherwise.

Remark: If $\mathcal{K}$ is bounded with $m = 0$, then the space ∂T_d contains a dense subset of points which are fixed only by the identity element.

Examples

Theorem (Lukina JLMS 2019)

Let p and d be distinct odd primes, let K be a finite unramified extension of the p-adic numbers $\mathbb{Q}_p$, and let

$$f(x) = (x+p)^d - p.$$

Then the arboreal representation $\rho_{f,0}$ is stable.

Theorem (Lukina JLMS 2019)

Suppose the image of an arboreal representation G_∞ is a subgroup of finite index in $Aut(T_d)$. Then the arboreal representation $\rho_{f,\alpha}$ is wild.

Further types of wild actions were defined in **Hurder and Lukina 2019**:

Types of wild actions

Let $(\partial T_d, G_\infty)$ be a Cantor action, and suppose $\mathcal{K}$ is unbounded.

1. The action $(\partial T_d, G_\infty)$ is *algebraically stable* if $\mathcal{Z}$ is bounded.
2. The action $(\partial T_d, G_\infty)$ is *wild of finite type* if for all $n \geq 0$ the groups K_n are finite.
3. The action $(\partial T_d, G_\infty)$ is *wild of flat type* if for all $n \geq 0$ we have the equality $Z_n = K_n$.
4. The action $(\partial T_d, G_\infty)$ is *dynamically wild* if for all $n \geq 0$ the inclusions $Z_n \subset K_n$ are proper.

More examples

Let t be a transcendental element, and let $K(t)$ be a field of functions.

Let $f(x)$ be a polynomial of degree $d \geq 2$ with coefficients in K.

Repeating a similar construction as before for the solutions of $f^n(x) = t$, we obtain

the arithmetic iterated monodromy group $\mathrm{Gal}_{\mathrm{arith}}(f) \subset Aut(T_d)$

and

the geometric iterated monodromy group $\mathrm{Gal}_{\mathrm{geom}}(f) \trianglelefteq \mathrm{Gal}_{\mathrm{arith}}(f)$.

For $d = 2$, the properties of these groups was described in **Pink 2013**.

Recall that a quadratic polynomial $f(x)$ is *post-critically finite*, if the orbit P_c of its critical point c under forward iterations of f is finite.

If the orbit P_c is finite, it may be periodic or pre-periodic.

The following theorems were proved in **Lukina 2018**:

Theorem 1 - geometric iterated monodromy group

Let $f(x)$ be a quadratic polynomial, $K = \mathbb{Q}$, and $\mathrm{Gal}_{\mathrm{geom}}(f)$ be the geometric iterated monodromy group acting on the binary tree T_2.

1. If P_c is infinite, then the action of $\mathrm{Gal}_{\mathrm{geom}}(f)$ is wild.
2. If $\#P_c = 1$, then the action of $\mathrm{Gal}_{\mathrm{geom}}(f)$ is stable with trivial discriminant group.
3. If $\#P_c \geq 2$ and the post-critical orbit is periodic, then the action of $\mathrm{Gal}_{\mathrm{geom}}(f)$ is wild.
4. If $\#P_c = 2$ and the the post-critical orbit is pre-periodic, then the action of $\mathrm{Gal}_{\mathrm{geom}}(f)$ is stable with finite discriminant group.
5. If $\#P_c \geq 3$ and the the post-critical orbit is pre-periodic, then the action of $\mathrm{Gal}_{\mathrm{geom}}(f)$ is wild.

Theorem 2 - arithmetic iterated monodromy group

Let $f(x)$ be a quadratic polynomial, $K = \mathbb{Q}$, and $\mathrm{Gal}_{\mathrm{arith}}(f)$ be the arithmetic iterated monodromy group acting on the binary tree T_2.

1. If P_c is infinite, then the action of $\mathrm{Gal}_{\mathrm{arith}}(f)$ is wild.
2. If $\#P_c = 1$, then the action of $\mathrm{Gal}_{\mathrm{arith}}(f)$ is stable with profinite discriminant group.
3. If $\#P_c \geq 2$ and the post-critical orbit is periodic, then the action of $\mathrm{Gal}_{\mathrm{arith}}(f)$ is wild.
4. If $\#P_c = 2$ and the the post-critical orbit is pre-periodic, then the action of $\mathrm{Gal}_{\mathrm{arith}}(f)$ is stable with profinite discriminant group.
5. If $\#P_c \geq 3$ and the the post-critical orbit is pre-periodic, then the action of $\mathrm{Gal}_{\mathrm{arith}}(f)$ is wild.

Recall that the action $(\partial T_2, G_\infty)$ is *dynamically wild* if for all $n \geq 0$ the inclusions $Z_n \subset K_n$ are proper.

Lukina 2018: If $\# P_c \geq 3$ and the post-critical orbit is pre-periodic, then $\mathrm{Gal}_{\mathrm{geom}}(f)$ and $\mathrm{Gal}_{\mathrm{arith}}(f)$ contain *non-Hausdorff elements*.

Hurder and Lukina 2019: If the profinite group G_∞ contains a non-Hausdorff element, then its action on the path space ∂T_d of the tree T_d is dynamically wild.

Corollary

1. If $\# P_c \geq 3$ and the post-critical orbit is pre-periodic, then the actions of $\mathrm{Gal}_{\mathrm{geom}}(f)$ and $\mathrm{Gal}_{\mathrm{arith}}(f)$ are dynamically wild.
2. If G_∞ has finite index in $Aut(T_d)$, then the action of G_∞ on T_d is dynamically wild.

In **Hurder and Lukina TAMS 2019**, we constructed examples of actions of torsion-free finite index subgroups of $\mathbf{SL}(n, \mathbb{Z})$ which are wild of flat type and of finite type.

Work in progress joint with **Álvarez López, Barral Lijo and Nozawa** constructs examples of actions which are a) dynamically wild of finite type, b) wild and algebraically stable.

We do not know if these actions may be realized as actions of arboreal representations associated to a polynomial.

Open problem

Find examples of arboreal representations whose actions are a) wild of finite type, b) wild of flat type, c) wild and algebraically stable.

References

- **Hurder and Lukina TAMS 2019:** Wild solenoids, Trans. Amer. Math. Soc., 371(7), 4493-4533.
- **Lukina JLMS 2019:** Arboreal Cantor actions, J. Lond. Math. Soc., 99(3) 2019, 678-706.
- **Lukina 2018:** Galois groups and Cantor actions, arXiv: 1809.08475, submitted.
- **Hurder and Lukina 2019:** Limit group invariants for non-free Cantor actions, arXiv:1904.11072, submitted.

Thank you for your attention!