Misiurewicz polynomials and irreducibility

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Presentation Outline

Post-critically finite polynomials

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When the set O_f is finite, we say **f(x) is post-critically finite (PCF)**. Post-critically finite polynomials are rather rare, so they are somewhat exceptional.

An example

Consider $f(x) = x^2 + i \in \mathbb{C}[x]$. It has the unique critical point 0.

We have the following orbit behavior:

$$\mathrm{i} \longrightarrow i-1 \rightleftarrows -i$$

So, O_f becomes

$$O_f = \{i, i-1, -i\},$$

and f is PCF.

A special family

We focus on a particular family, namely the PCF polynomials of the form $f_c(x) = x^2 + c \in \mathbb{C}[x]$. The post-critical orbit becomes

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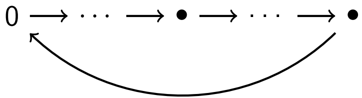
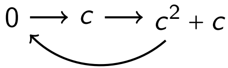
In particular, any c_0 value for which f_{c_0} is PCF is an algebraic integer.

Periodic case



Example: Let's find c values such that 0 is periodic under f_c with exact period 3.



$$f_c^3(0) = (c^2 + c)^2 + c = c(c^3 + 2c^2 + c + 1) = 0$$

$\implies$ For any root c_0 of $c^3 + 2c^2 + c + 1$, f_{c_0} will have 0 as a periodic point of exact period 3.

Gleason polynomials

Definition: The **Gleason polynomial** G_n is the factor of $f_c^n(0)$ whose roots are the c_0 values for which 0 is periodic under f_{c_0} with exact period n.

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$\boldsymbol{G_1} = c.$

$\boldsymbol{G_2} = c + 1.$

$\boldsymbol{G_3} = c^3 + 2c^2 + c + 1.$

$\boldsymbol{G_4} = c^6 + 3c^5 + 3c^4 + 3c^3 + 2c^2 + 1.$

$\boldsymbol{G_5} = c^{15} + 8c^{14} + 28c^{13} + 60c^{12} + 94c^{11} + 116c^{10} + 114c^9 + 94c^8 + 69c^7 + 44c^6 + 26c^5 + 14c^4 + 5c^3 + 2c^2 + c + 1.$

Gleason polynomials

Conjecture: For any $n \geq 1$, G_n is irreducible over $\mathbb{Q}$.

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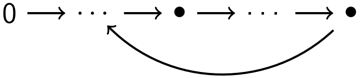
If true, this would be a dynamical analog of the well-known fact that cylotomic polynomials are irreducible over $\mathbb{Q}$. **But**, it looks difficult.

Strictly preperiodic case



f_c has orbit type (m, n) when $f_c^m(0)$ is the first periodic element, and n is the exact period (so $f_c^m(0) = f_c^{m+n}(0)$).

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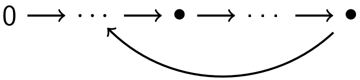

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Definition: The **Misiurewicz polynomial** $G_{m,n}$ is the factor of $f_c^{m+n}(0) - f_c^m(0)$ whose roots are the c_0 values for which 0 is strictly preperiodic under f_{c_0} with orbit type (m, n).

Some Misiurewicz polynomials

$$\boldsymbol{G_{2,2}} = c^2 + 1.$$

$$\boldsymbol{G_{2,3}} = c^6 + 2c^5 + 2c^4 + 2c^3 + c^2 + 1.$$

$$\boldsymbol{G_{2,4}} = c^{12} + 6c^{11} + 15c^{10} + 22c^9 + 23c^8 + 18c^7 + 11c^6 + 8c^5 + 6c^4 + 2c^3 + 1.$$

$$\boldsymbol{G_{2,5}} = c^{30} + 14c^{29} + 92c^{28} + 384c^{27} + 1164c^{26} + 2768c^{25} + 5412c^{24} + 8964c^{23} + 12854c^{22} + 16236c^{21} + 18316c^{20} + 18676c^{19} + 17394c^{18} + 14912c^{17} + 11834c^{16} + 8730c^{15} + 6001c^{14} + 3862c^{13} + 2344c^{12} + 1348c^{11} + 738c^{10} + 384c^9 + 190c^8 + 90c^7 + 41c^6 + 18c^5 + 6c^4 + 2c^3 + c^2 + 1.$$

Presentation Outline

The main question

Question: Is it true that the polynomial $G_{m,n}$ is always irreducible over $\mathbb{Q}$?

Presentation Outline

A summary of known results

Theorem (G; 2018)

For any $m \neq 0$, $G_{m,1}$ and $G_{m,2}$ are irreducible over $\mathbb{Q}$.

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For any $m \neq 0$, $G_{m,1}$ *and* $G_{m,2}$ *are irreducible over* $\mathbb{Q}$.

Theorem (Buff, Epstein, Koch; 2018)

For any $m \neq 0$, $G_{m,3}$ *is irreducible over* $\mathbb{Q}$.

Presentation Outline

A new result

Theorem (G; 2019)

For any $m \neq 0$, the number of irreducible factors of $G_{m,n}$ over $\mathbb{Q}$ is bounded from above by the number of irreducible factors of the reduced polynomial $\overline{G_n} \in \mathbb{F}_2[c]$. In particular, if the reduced polynomial $\overline{G_n} \in \mathbb{F}_2[c]$ is irreducible, then $G_{m,n}$ is irreducible over $\mathbb{Q}$.

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- It recovers the previously known results ($n = 1, 2, 3$).
- It gives an upper bound on the number of irreducible factors of any Misiurewicz polynomial, which depends only on the period.
- It simplifies the general irreducibility question by reducing it to eliminating some particular possibilities.

One particular example

Take $n = 4$. It is not known whether $G_{m,4}$ is irreducible over $\mathbb{Q}$ for all $m \neq 0$ or not.

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One particular example

Take $n = 4$. It is not known whether $G_{m,4}$ is irreducible over $\mathbb{Q}$ for all $m \neq 0$ or not. By the theorem, it could have at most 2 irreducible factors over $\mathbb{Q}$. Suppose it is reducible for some $m \neq 0$. It follows from the proof of the theorem that we must then have

$$G_{m,4} = [(c^2 + c + 1)^{c_m} + 2f(c)][(c^4 + c + 1)^{c_m} + 2g(c)],$$

where $c_m = 2^{m-1}$ if $m \not\equiv 1 (\text{mod } 4)$, and $c_m = 2^{m-1} - 1$ otherwise.

Some notation

Let $m \neq 0$, and c_0 a root of $G_{m,n}$. The post-critical orbit of f_{c_0} becomes

$$\{a_1, a_2, \ldots, a_{m+n-1}\},$$

where $a_i = f^i_{c_0}(0)$. Set $K = \mathbb{Q}(c_0)$, and let $\mathcal{O}_K$ be its ring of integers.

For $a \in \mathcal{O}_K$, we will denote by (a) the ideal of $\mathcal{O}_K$ generated by a.

A phenomenon in the critical orbit

Theorem (G; 2018)

Let $f_{c_0}(x) = x^2 + c_0 \in \bar{\mathbb{Q}}[x]$ be a PCF polynomial having orbit type (m, n) with $m \neq 0$. Set $K = \mathbb{Q}(c_0)$, and let $O_{f_{c_0}} = \{a_1, a_2, \dots, a_{m+n-1}\} \subset \mathcal{O}_K$ be the critical orbit of f_{c_0}. Then the following holds:

If $n \nmid i$, then a_i is a unit.

If $n \mid i$, then one has $(a_i)^{M_{m,n}} = (2)$, where

$$M_{m,n} = \begin{cases} 2^{m-1} & \text{if } n \nmid m-1 \\ 2^{m-1} - 1 & \text{if } n \mid m-1. \end{cases}$$

Ideas from the proof

Let c_0 be a root of $G_{m,n}$, and set $K = \mathbb{Q}(c_0)$.

Strategy: Ideally; find $[K : \mathbb{Q}]$ by studying the ramification of (2) in K, and then compare it with $\deg(G_{m,n})$.

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- The theorem in the previous slide directly implies the irreducibility of $G_{m,n}$ when $n = 1, 2$.
- For $n > 2$, one needs a more refined ramification information about the ideal (2). For this, I found a factorization of the ideal (a_n) in $\mathcal{O}_K$.

THANK YOU!!

Extra - ideas from the proof

Let $\overline{G_n} = f_1 \cdots f_k \in \mathbb{F}_2[c]$, and let $\tilde{f}_1, \ldots, \tilde{f}_k \in \mathbb{Z}[c]$ be any lifts of $f_1, \ldots, f_k$, respectively.

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Proposition (G; 2019)

Let $f_{c_0}(x) = x^2 + c_0 \in \bar{\mathbb{Q}}[x]$ be a PCF polynomial having orbit type (m, n) with $m \neq 0$. Set $K = \mathbb{Q}(c_0)$, and let $O_{f_{c_0}} = \{a_1, a_2, \ldots, a_{m+n-1}\} \subset \mathcal{O}_K$ be the critical orbit of f_{c_0}. Then the following holds:

$$(a_n) = (2, \tilde{f}_1(c_0)) \cdots (2, \tilde{f}_k(c_0)).$$