A gap theorem in arithmetic dynamics

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Problem

We assume we have the following data:

- A quasi-projective variety X defined over a number field;
- A dominant rational self-map $\Phi : X \to X$;
- A rational map $f : X \to \mathbb{P}^1$;
- A point $c \in X(\bar{\mathbb{Q}})$.

We are interested in the quantity

$$h(f(\Phi^n(c))),$$

where h is the logarithmic Weil height.

Of course we also assume that the orbit of c under Φ avoids the indeterminacy locus of f and Φ so that everything makes sense.

One can think of this intuitively as we are applying f (something "like" a linear functional or a measurement) to the elements in the orbit of c under Φ and trying to understand the complexity of the resulting sequence.

Why should one care?

Many classical sequences from number theory and algebraic combinatorics fall under this framework. Let's look at some examples.

Factorials! (Yes, the exclamation point is intentional.)

Let $X = \mathbb{A}^2$ and let $\Phi(x, y) = (x + 1, yx)$ and let f be projection onto the second coordinate. Notice that $\Phi^n(1, 1) = (n + 1, n!)$ and so $f(\Phi^n(1, 1)) = n!$

Fibonacci numbers.

Let $X = \mathbb{A}^2$ and let $\Phi(x, y) = (y, x + y)$ and let f be projection onto the first coordinate. Notice that $\Phi^n(1, 1) = (F_n, F_{n+1})$ and so $f(\Phi^n(1, 1))$ is the n-th Fibonacci number.

Catalan numbers.

You might recall that the n-th Catalan number counts walks from $(0,0)$ to (n,n) that never go below the main diagonal.

Let $X = \mathbb{A}^2$ and let $\Phi(x,y) = (x+1, 2y(2x+1)/(x+2))$ and let f be projection onto the second coordinate. Exercise: $f(\Phi^n(1,1))$ is the n-th Catalan number.

FACT: every sequence whose generating series satisfies a homogeneous linear differential equation with rational function coefficients can be realized in this way. Such sequences are called *holonomic sequences* or D-*finite sequences* and they are ubiquitous in number theory and algebraic combinatorics (see Chapter 6 of Stanley's *Enumerative Combinatorics, vol. 2*).

The proof of the fact appears in chapter three in the book I have with Dragos and Tom.

Thus we see that the framework above gives a dynamical extension of holonomic sequences.

There is a long history of trying to extract asymptotic information from such sequences. (One can see, for example, the book of Flajolet and Sedgewick.) Since our sequences are not in general integer- or even real-valued, the asymptotics are less useful in our setting. We can, however, use the height as a measure of complexity and in the positive-integer-valued case it gives the same type of coarse asymptotic information as before.

van der Poorten-Shparlinski proved a "gap" result.

Theorem: (vdP-S, 1996) If $F(x) = \sum a_n x^n \in \mathbb{Q}[[x]]$ satisfies a homogeneous linear differential equation with rational function coefficients. Then if $h(a_n)/\log\log(n) = o(1)$ then the sequence a_n is eventually periodic.

We can interpret this as saying that the coefficients must have at least some minimal complexity unless they are eventually periodic.

This was proved using a type of Schanuel's result (bounding points of a given height in some projective space) along with a pigeonhole principle and a recurrence one obtains from the differential equation.

Last year, B-Nguyen-Zannier strengthened this result and showed that for $F(x) = \sum a_n x^n \in \bar{\mathbb{Q}}[[x]]$ satisfying a homogeneous linear differential equation with rational function coefficients. Then if $h(a_n)/\log(n) = \mathrm{o}(1)$ then the sequence a_n is eventually periodic.

In particular, this replaces the $\log\log(n)$ in the vdP-S result with $\log(n)$. This appears to be close to best possible:

$F(x) = \log(1+x) = x - x^2/2 + x^3/3 - \cdots$ satisfies a differential equation and the height of the coefficient of x^n is $\log(n)$, so we cannot push it too much further.

I should point out that B-Nguyen-Zannier gave a result for multivariate holonomic series (where one needs a suitable notion of “eventually periodic”), but that’s a story for another time....

In light of this gap, it’s natural to ask whether a similar gap holds in the dynamical framework given above.

We were able to prove a stronger dynamical gap result under the condition that our map is an étale endomorphism.

Fibre Bound Theorem: Let X and Y be quasi-projective varieties defined over a field K of characteristic 0, let $f : X \to Y$ be a morphism defined over K, let $\Phi : X \to X$ be an étale endomorphism, and let $c \in X(K)$. If $f(\Phi^n(c))$ is not eventually periodic then there is a fixed N (independent of α) such that for each $\alpha \in Y$, the number of n such that $f(\Phi^n(c)) = \alpha$ is at most N.

The proof uses the p-adic interpolation method along with estimates for the number of zeros of p-adic analytic functions.

As a corollary (using bounds on the number of points of a given height in projective space), we get that under the hypotheses from the theorem and with K a number field, either $f(\Phi^n(c))$ is eventually periodic or

$$\limsup h(f(\Phi^n(c)))/\log(n)$$

is strictly positive. Let's call this, for the purposes of this talk, the *Gap Theorem*.

Strengthening the Fibre Bound Theorem outside of the étale endomorphism case is hard: one could deduce the Dynamical Mordell-Lang theorem from our theorem if one knew it held for general endomorphisms. Also, the conclusion doesn't even hold in positive characteristic.

It is completely reasonable, on the other hand, to expect that the hypotheses in the Gap Theorem can be relaxed.

Fei Hu, Matt, and I have been working on strengthening the Gap Theorem. So far we have proved that under the weaker general hypotheses from the original set-up either $f(\Phi^n(c))$ is eventually periodic or

$$\limsup h(f(\Phi^n(c)))/\log(n)$$

is strictly positive.

In fact, we believe much more should hold: if

$$\limsup h(f(\Phi^n(c)))/\log(n)$$

is strictly positive, then there is a fixed constant $C > 0$ such that outside of a set of n of zero density we have

$$h(f(\Phi^n(c))) > C\log(n).$$

We pretty much believe we can prove this, but we have to work through the details.

One might ask whether this zero density set can be replaced by a finite set. The answer is ‘no’ if one works with function fields in positive characteristic.

If one could do this in characteristic zero, one could prove DML! I take that to mean that going from zero density to finite is probably not going to happen anytime soon.

Thanks!