A Transcendental Dynamical Degree

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Joint work with Jason Bell and Mattias Jonsson.

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- In homogeneous coordinates $f = [F_0 : F_1 : F_2]$ where $F_j(x_0, x_1, x_2)$ are homogeneous polynomials with (the same) degree $\deg(f)$ and no non-constant common factors.

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- Note $\deg(f^{n+m}) \leq (\deg f^n)(\deg f^m)$ for any $n, m \in \mathbf{N}$.
- Hence (Russakovski-Shiffman) can define the *dynamical degree*

$$\lambda(f) := \lim_{n\to\infty} (\deg f^n)^{1/n}.$$

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- If $f = f_A = (x^a y^b, x^c y^d)$ is monomial, then $\lambda(f)$ is the spectral radius of $A := \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, i.e. a quadratic integer.

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- D-Favre: If $f : \mathbf{P}^2 \to \mathbf{P}^2$ is birational, then $\lambda(f)$ is an algebraic integer.

Known cases (cont)

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This is not necessarily true for monomial maps.

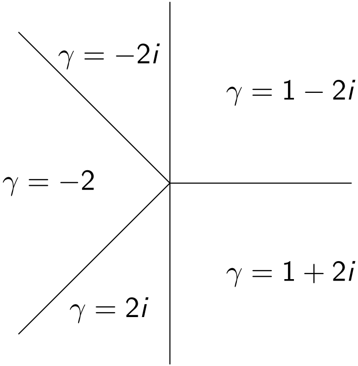
Theorem (Hasselblatt-Propp, Favre)

Let $\zeta \in \mathbf{Z}[i]$ be a Gaussian integer such that $\zeta^n \notin \mathbf{R}$ for any $n \in \mathbf{N}$ and $A = \begin{bmatrix} \mathrm{Re}\,\zeta & -\mathrm{Im}\,\zeta \\ \mathrm{Im}\,\zeta & \mathrm{Re}\,\zeta \end{bmatrix}$. Then the degree sequence $(\deg f_A^n)_{n\in\mathbf{N}}$ does not satisfy a linear recursion relation.

Degrees of monomial maps

Let $\Gamma = \{-2, \pm 2i, 1 \pm 2i\}$. Then

$$\deg(f_A^n) = \max_{\gamma \in \Gamma} \mathrm{Re}\, \gamma \zeta^n.$$



Transcendental example

Let $\tau = f_A$ be the monomial map corresponding to the Gaussian integer ζ above and $\sigma : \mathbf{P}^2 \to \mathbf{P}^2$ be the birational involution given in affine coordinates by

$$\sigma(x,y) = \left(x\frac{x-y-1}{x+y-1}, y\frac{y-x-1}{x+y-1} \right).$$

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First step of proof: toric geometry gives

Proposition

$\lambda(f) \in (\lambda(\tau), \infty)$ *is the unique positive solution of*

$$\sum_{n \geq 1} \frac{\deg \tau^j}{\lambda(f)^j} = 1.$$

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Main idea: if $n\theta$ is nearly an integer, then $\gamma(j)$ is nearly n-periodic in j. Hence $F(z)$ is nearly, but not exactly, equal to the rational function

$$F_n(z) := \frac{1}{1 - z^n} \sum_{j=1}^{n} \gamma(j) z^n.$$

Proof of transcendence (cont)

Lemma

For all $n \in \mathbf{N}$, *we have* $1 = \operatorname{Re} F(\alpha) > \operatorname{Re} F_n(\alpha)$.

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Set

$$E_n(z) := |1 - z^n|^2 \mathrm{Re}\,(F(z) - F_n(z)) = \mathrm{Re}\,(1 - \bar{z})^n \sum_{j>n} (\gamma(j) - \gamma(j-n)) z^j.$$

Then because $\mathrm{Re}\, F(\alpha) = 1$, we have that $E_n(\alpha)$ is a non-zero polynomial in $\alpha, \bar{\alpha}$ with degree $2n$ and coefficients in the (n-independent) finite set $\Gamma \subset \bar{\mathbf{Q}}$.

Proof of Transcendence (cont)

By a Diophantine approximation theorem of Evertse (descendent of the p-adic version of the Schmitt Subspace Theorem), we then get

Corollary

There exists $\delta > 0$ *such that* $|E_n(\alpha)| > \delta^{2n}$ *for any* $n \in \mathbf{N}$.

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Definition

An index $j > n$ is *n-irregular* if $\gamma(j) \neq \gamma(j - n)$.

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For such n, we have $|E_n(\alpha)| \leq M|\alpha|^{Cn}$ for some M independent of n, contrary (for large C and n) to our previous bound.

If θ is badly approximable and m/n is a continued fraction approximation of θ, then we can still control the number and spacing of n-irregular indices $j \in (n, Cn]$.

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$$E_n(\alpha) + \mathrm{Re}\,(1 - \bar{\alpha}^n) \sum_{j \in (n, Cn]} (\gamma(j) - \gamma(j-n))\alpha^n.$$

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Crucially, δ does not depend on C.

Conclusion

We have an upper bound $M|\alpha|^{Cn}$ as in the well-approximable case. Upper and lower bounds again conflict for large C and n.

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Thank You!