Type I Repelling Density for Non-archimedean Polynomials (Work in Progress)

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Julia and Filled Julia Sets

Throughout, $f(z) \in \mathbb{C}_p[z]$ will (usually) be a polynomial of degree ≥ 2.

Definition

The *filled Julia set* $\mathcal{K}$ of f is the set of points that stay bounded under iteration:

$$\mathcal{K} := \left\{ x : \limsup_{n\to\infty} |f^n(x)| < \infty \right\}.$$

This definition works whether the ambient space is the field $\mathbb{C}_p$ or the Berkovich space $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$. We'll write $\mathcal{K}_{\mathrm{I}}$ for the "type I set", i.e. the filled Julia points in $\mathbb{C}_p$.

Definition

The *Julia set* for a polynomial is the boundary of its filled Julia set $\mathcal{J} := \partial\mathcal{K}$. We'll write $\mathcal{J}_{\mathrm{I}} := \mathcal{J} \cap \mathbb{C}_p$ for the type I points. For polynomials, $\mathcal{J}$ is always either a single point or a topological Cantor set.

Repelling Density in Complex Dynamics

One of the earliest and most fundamental theorems in complex dynamics is that the repelling periodic points are dense in the Julia set of a rational function. A natural question (first considered by Hsia), is whether this also holds for non-archimedean rational functions.

Fatou and Julia each gave a proof in the complex setting, and these different methods both provide insight in the non-archimedean setting.

Fatou's Strategy

- Start with an arbitrary open set U intersecting $\mathcal{J}_{\mathbb{C}}$.
- (After possibly shrinking U) construct analytic inverses g_1, g_2 of f on U with disjoint images.
- If $f^n(z) - z$ has no roots in U for any n, then the rational functions

$$\frac{f^n - g_1}{f^n - g_2} \cdot \frac{z - g_2}{z - g_1}$$

form a normal family on U, contradicting that $\{f^n\}$ do not form a normal family on U.
- Thus, $\mathcal{J}_{\mathbb{C}}$ is contained in the closure of the set of all periodic points, but in complex dynamics the number of non-repelling cycles is bounded.

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Fatou's Strategy

The last bullet point is false in non-archimedean dynamics! For example, all periodic points of $f(z) = z^2$ are non-repelling. But following this strategy, does give the following result:

Theorem (Hsia, 2000)

The type I Julia set $\mathcal{J}_I$ is contained in the closure of all type I periodic points of f.

A refinement of this argument dealing with preimages in Berkovich space also gives an alternate proof of the following result (see Benedetto's new book!):

Theorem (Rivera-Letelier)

The Berkovich Julia set $\mathcal{J}$ is the closure of all repelling Berkovich periodic points of f.

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Julia's Strategy

- Start with a repelling periodic point x_0 (WLOG fixed point).
- Construct a *homoclinic* backward orbit

$$\cdots x_2 \mapsto x_1 \mapsto x_0 \qquad \lim_{n\to\infty} x_n = x_0$$

- Find a periodic point in a small neighborhood of some x_n that must necessarily be repelling because x_0 is repelling.

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Julia's Strategy

Here, the first bullet point fails (for type I points) in the non-archimedean setting (again consider $f(z) = z^2$), but following this strategy gives the following result:

Theorem (Bézivin, 2001)

Assume f has at least one type I repelling periodic point. Then, the type I repelling periodic points are dense in $\mathcal{J}_I$.

Rivera-Letelier also used this strategy for his original proof of the density of all Berkovich repelling periodic points.

Statement of Conjecture

Conjecture (Work in Progress)

Suppose $f \in \mathbb{C}_p[z]$ has no type I repelling periodic points. Then, its Berkovich Julia set $\mathcal{J}$ satisfies

$$\inf_{x \in \mathcal{J}} diam(x) > 0.$$

In particular, $\mathcal{J}_I = \emptyset$.

The infimum condition above can be restated purely in terms of type I points:

- There exists $r > 0$ such that for every $\alpha \in \mathcal{K}_{\mathrm{I}}$, the closed disc of radius r about α is also in the filled Julia set.

Good Reduction

Example

If f has explicit good reduction, e.g. $f(z) = z^2$, $f(z) = z^3 + pz$, then there are no type I repelling periodic points and

- $\mathcal{J} = \{\zeta_{0,1}\}$
- $\mathcal{J}_{\mathrm{I}} = \emptyset$
- $\inf_{x \in \mathcal{J}} \operatorname{diam}(x) = 1$.

Theorem (Benedetto)

Assume $f \in \mathbb{C}_p[z]$ has degree $\leq p+1$. Then, f has no type I repelling periodic points if and only if f has (potentially) good reduction.

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Examples

Example (Rivera-Letelier)

Consider $f(z) = (z^p - z^{p^2})/p$.

- $f(z)$ does not have (potentially) good reduction.
- $\mathcal{K} \subseteq \overline{D}(0,1)$.
- $|f'(z)| = \left| z^{p-1} - pz^{p^2-1} \right| \le 1$ on $\mathcal{K}$.
- Every point $x \in \mathcal{J}$ satisfies $\operatorname{diam}(x) = |p|^{1/(p-1)}$.

Question: Is $|f'(z)| \le 1$ on the filled Julia set whenever f has no type I repelling periodic points?

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Derivative Bound

...No (see next example), but it's not too hard to show that the Conjecture is true if this bound does hold. In fact, to prove the Conjecture you only need the following:

Lemma (Conjecture/Work in Progress)

Assume f has no type I repelling periodic points. Then, there is a constant $C = C(f)$ such that

$$\bigl|(f^n)'(\alpha)\bigr| \le C$$

for all $n \ge 1$ and $\alpha \in \mathcal{K}_I$.

Examples

Example (Benedetto)

Assume $p \geq 3$. Choose $a \in \mathbb{C}_p$ with absolute value $r := |a| = |p|^{-e} > 1$, where $e > 0$. Consider $f(z) = z^2(z-a)^p$.

- $f(z)$ does not have (potentially) good reduction.
- $\mathcal{K} \subset \overline{D}(0, r^{(1-p)/2}) \cup \overline{D}(a, r^{-1/p})$.
- $|f'(z)| = r^p\,|z| > 1$ on the annulus $\overline{D}(0, r^{(1-p)/2}) \setminus \overline{D}(0, r^{-p})$, and this annulus does contain points of $\mathcal{K}_{\mathrm{I}}$.
- The following are equivalent:
  - $f(z)$ has no type I repelling periodic points.
  - $|(f^n)'(\alpha)| \leq r^{(p+1)}$ for all $n \geq 1$ and $\alpha \in \mathcal{K}_{\mathrm{I}}$.
  - $e \leq p/(2p+2)$.

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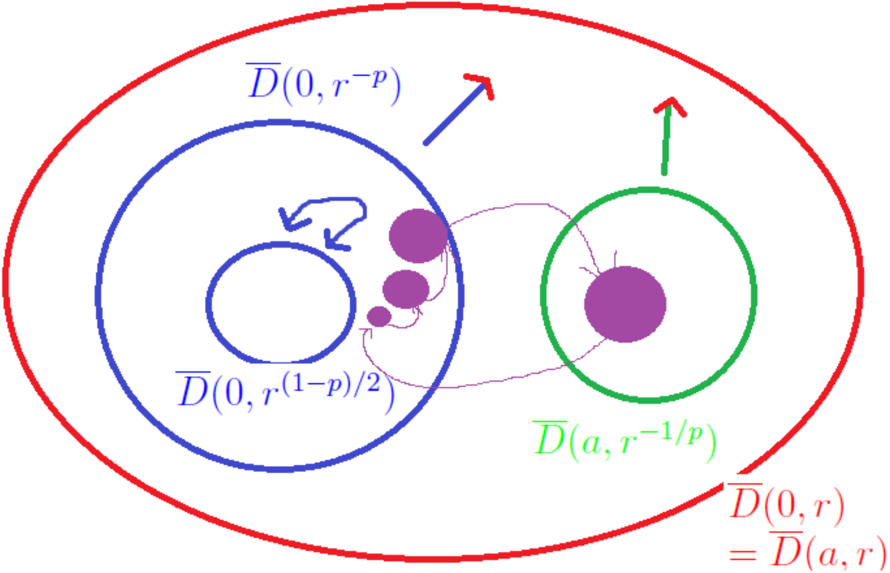
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Stability

Proposition

Let $f_\lambda(z) \in \mathbb{C}_p\langle\lambda\rangle[z]$ *be an analytic family of polynomials parameterized by the closed unit disc. Assume the Conjecture holds for* $f_0(z)$*. Then, there exists* $0 < s < 1$ *such that* $f_\lambda(z)$ *has the same Julia set as* $f_0(z)$ *for all* $\lambda \in \overline{D}(0, s)$*.*

Corollary

Assume the Conjecture holds for $f \in \mathbb{C}_p[z]$*. Then,* f *has no wandering domains.*

Proof of Corollary.

Adjust the coefficients of f so that they lie in $\overline{\mathbb{Q}_p}$. By the Proposition, this can be done without altering the Julia set of f. A theorem of Benedetto implies that this new polynomial has no wandering domains. □

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