Dynamical height growth: left, right, and total orbits.

Wade Hindes

(Texas State University)

Joint Mathematics Meetings 2020

Denver, CO.

January 17, 2020.

Notation:

- $S = \{\phi_1, \ldots, \phi_s\}$ is a set of dominant, rational self-maps on $\mathbb{P}^N$ defined over $\overline{\mathbb{Q}}$.
- M_S is the monoid (semigroup) generated by S under composition.
- $\mathbb{P}^N(\overline{\mathbb{Q}})_S = \big\{P \in \mathbb{P}^N(\overline{\mathbb{Q}}) : f(P)$ is defined for all $f \in M_S\big\}$.

Notation:

- $S = \{\phi_1, \ldots, \phi_s\}$ is a set of dominant, rational self-maps on $\mathbb{P}^N$ defined over $\overline{\mathbb{Q}}$.
- M_S is the monoid (semigroup) generated by S under composition.
- $\mathbb{P}^N(\overline{\mathbb{Q}})_S = \big\{P \in \mathbb{P}^N(\overline{\mathbb{Q}}) : f(P) \text{ is defined for all } f \in M_S\big\}$.

Goals:

1. Study the arithmetic properties of the dynamical system(s) generated by S.
2. Generalize known problems from when S is a singleton.

Notation:

- $S = \{\phi_1, \ldots, \phi_s\}$ is a set of dominant, rational self-maps on $\mathbb{P}^N$ defined over $\overline{\mathbb{Q}}$.
- M_S is the monoid (semigroup) generated by S under composition.
- $\mathbb{P}^N(\overline{\mathbb{Q}})_S = \big\{P \in \mathbb{P}^N(\overline{\mathbb{Q}}) : f(P) \text{ is defined for all } f \in M_S\big\}$.

Goals:

1. Study the arithmetic properties of the dynamical system(s) generated by S.
2. Generalize known problems from when S is a singleton.

Examples: dynamical degrees, canonical heights, arboreal reps., integral points, primitive primes, DML, . . .

Remark: To make these problems make sense, we need to consider various types of orbits: sequential and total.

Remark: To make these problems make sense, we need to consider various types of orbits: sequential and total.

- The *total orbit* of $P \in \mathbb{P}^N(\overline{\mathbb{Q}})_S$ is the set

$$\mathrm{Orb}_S(P) := \{f(P) : f \in M_S\}.$$

Remark: To make these problems make sense, we need to consider various types of orbits: sequential and total.

- The *total orbit* of $P \in \mathbb{P}^N(\overline{\mathbb{Q}})_S$ is the set

$$\mathrm{Orb}_S(P) := \{f(P) : f \in M_S\}.$$

- Given a sequence $\gamma = (\theta_1, \theta_2, \dots)$ with $\theta_i \in S$, set

$$\gamma_n^- := \theta_n \circ \theta_{n-1} \circ \cdots \circ \theta_1$$

and

$$\gamma_n^+ := \theta_1 \circ \theta_2 \circ \cdots \circ \theta_n.$$

Remark: To make these problems make sense, we need to consider various types of orbits: sequential and total.

- The *total orbit* of $P \in \mathbb{P}^N(\overline{\mathbb{Q}})_S$ is the set

$$\mathrm{Orb}_S(P) := \{f(P) : f \in M_S\}.$$

- Given a sequence $\gamma = (\theta_1, \theta_2, \dots)$ with $\theta_i \in S$, set

$$\gamma_n^- := \theta_n \circ \theta_{n-1} \circ \cdots \circ \theta_1$$

and

$$\gamma_n^+ := \theta_1 \circ \theta_2 \circ \cdots \circ \theta_n.$$

Then the *left and right γ-orbits* of $P \in \mathbb{P}^N(\overline{\mathbb{Q}})_S$ are

$$\mathrm{Orb}_\gamma^-(P) := \{\gamma_n^-(P)\}_{n \geqslant 0} \quad \text{and} \quad \mathrm{Orb}_\gamma^+(P) := \{\gamma_n^+(P)\}_{n \geqslant 0}$$

respectively.

Features of various types of orbits

Each type of orbit has different uses/features:

Features of various types of orbits

Each type of orbit has different uses/features:

Examples: (Vaguely)

- Arithmetic properties of $\mathrm{Orb}_S(P)$ for generic P can detect (in practice) relations between the maps in S (e.g., if they commute).

Features of various types of orbits

Each type of orbit has different uses/features:

Examples: (Vaguely)

- Arithmetic properties of $\mathrm{Orb}_S(P)$ for generic P can detect (in practice) relations between the maps in S (e.g., if they commute).
- For critical points P, the right orbits $\mathrm{Orb}_\gamma^+(P)$ control the ramification in some (generalized) dynamically generated Galois extensions.

Each type of orbit has different uses/features:

Examples: (Vaguely)

- Arithmetic properties of $\mathrm{Orb}_S(P)$ for generic P can detect (in practice) relations between the maps in S (e.g., if they commute).
- For critical points P, the right orbits $\mathrm{Orb}_\gamma^+(P)$ control the ramification in some (generalized) dynamically generated Galois extensions.
- Left orbits define a (type of) random walk in $\mathbb{P}^N$. What does it mean if these random walks return to a subvariety i.o. with positive probability?

Each type of orbit has different uses/features:

Examples: (Vaguely)

- Arithmetic properties of $\mathrm{Orb}_S(P)$ for generic P can detect (in practice) relations between the maps in S (e.g., if they commute).
- For critical points P, the right orbits $\mathrm{Orb}_\gamma^+(P)$ control the ramification in some (generalized) dynamically generated Galois extensions.
- Left orbits define a (type of) random walk in $\mathbb{P}^N$. What does it mean if these random walks return to a subvariety i.o. with positive probability?

Philosophy: One way to understand a subset T of $\mathbb{P}^N$ (like $T = \mathrm{Orb}_S(P), \dots$) is to find out many points T has.

More precisely, to understand the arithmetic of various orbits, we study the growth rates of the Weil height $h : \mathbb{P}^N(\overline{\mathbb{Q}}) \to \mathbb{R}$:

$$\lim_{B\to\infty} \#\big\{Q \in \mathrm{Orb}_\gamma^-(P) : h(Q) \leqslant B\big\} = ?$$

$$\lim_{B\to\infty} \#\big\{Q \in \mathrm{Orb}_\gamma^+(P) : h(Q) \leqslant B\big\} = ??$$

$$\lim_{B\to\infty} \#\big\{Q \in \mathrm{Orb}_S(P) : h(Q) \leqslant B\big\} = ???$$

More precisely, to understand the arithmetic of various orbits, we study the growth rates of the Weil height $h : \mathbb{P}^N(\overline{\mathbb{Q}}) \to \mathbb{R}$:

$$\lim_{B\to\infty} \#\big\{Q \in \mathrm{Orb}_\gamma^-(P) : h(Q) \leqslant B\big\} = ?$$

$$\lim_{B\to\infty} \#\big\{Q \in \mathrm{Orb}_\gamma^+(P) : h(Q) \leqslant B\big\} = ??$$

$$\lim_{B\to\infty} \#\big\{Q \in \mathrm{Orb}_S(P) : h(Q) \leqslant B\big\} = ???$$

A key fact (as in the case of a single function) is

$$\boxed{h(f(P)) \leqslant \deg(f)h(P) + C(f)},$$

along with the corresponding lower bound for morphisms.

More precisely, to understand the arithmetic of various orbits, we study the growth rates of the Weil height $h : \mathbb{P}^N(\overline{\mathbb{Q}}) \to \mathbb{R}$:

$$\lim_{B\to\infty} \#\left\{Q \in \mathrm{Orb}_\gamma^-(P) : h(Q) \leqslant B\right\} = ?$$

$$\lim_{B\to\infty} \#\left\{Q \in \mathrm{Orb}_\gamma^+(P) : h(Q) \leqslant B\right\} = ??$$

$$\lim_{B\to\infty} \#\left\{Q \in \mathrm{Orb}_S(P) : h(Q) \leqslant B\right\} = ???$$

A key fact (as in the case of a single function) is

$$\boxed{h(f(P)) \leqslant \deg(f)h(P) + C(f)},$$

along with the corresponding lower bound for morphisms.

Heuristic: $h(f(P))$ grows like $\deg(f)$ for "generic" P.

Dynamical degrees for sequences

In particular, to study arithmetic properties of orbits, we can try to understand the growth rate of degrees (as we iterate).

Dynamical degrees for sequences

In particular, to study arithmetic properties of orbits, we can try to understand the growth rate of degrees (as we iterate).

Remarks:

- The case of sequences $\gamma = (\theta_1, \theta_2, \dots)$ is more naturally analogous to the case of a single function.
- Philosophy: by sampling "enough" γ-orbits, we uncover properties of total orbits.

Dynamical degrees for sequences

In particular, to study arithmetic properties of orbits, we can try to understand the growth rate of degrees (as we iterate).

Remarks:

- The case of sequences $\gamma = (\theta_1, \theta_2, \dots)$ is more naturally analogous to the case of a single function.
- Philosophy: by sampling "enough" γ-orbits, we uncover properties of total orbits.

The degrees $\deg(\gamma_n^+)$ and $\deg(\gamma_n^-)$ tend to grow exponentially. With this in mind, define

$$\lim_{n\to\infty} \deg(\gamma_n^-)^{1/n} \quad \text{and} \quad \lim_{n\to\infty} \deg(\gamma_n^+)^{1/n},$$

called the *left and right dynamical degrees* of γ.

Warning!!! These limits may not exist in general.

Warning!!! These limits may not exist in general.

Example: Let $\phi_1, \phi_2 : \mathbb{P}^N \to \mathbb{P}^N$ be morphisms of degree $d_1 \neq d_2$, let $S = \{\phi_1, \phi_2\}$, and let

$$\gamma \leftrightarrow (1, 2, 2, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 1, 1, \dots);$$

Warning!!! These limits may not exist in general.

Example: Let $\phi_1, \phi_2 : \mathbb{P}^N \to \mathbb{P}^N$ be morphisms of degree $d_1 \neq d_2$, let $S = \{\phi_1, \phi_2\}$, and let

$$\gamma \leftrightarrow (1, 2, 2, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 1, 1, \dots);$$

Then, the limits defining the dynamical degrees do not exist:

Warning!!! These limits may not exist in general.

Example: Let $\phi_1, \phi_2 : \mathbb{P}^N \to \mathbb{P}^N$ be morphisms of degree $d_1 \neq d_2$, let $S = \{\phi_1, \phi_2\}$, and let

$$\gamma \leftrightarrow (1, 2, 2, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 1, 1, \dots);$$

Then, the limits defining the dynamical degrees do not exist:

$$\deg(\gamma^{-}_{2^k-1}) = \begin{cases} d_1^{\frac{2}{3}2^k - \frac{1}{3}} d_2^{\frac{1}{3}2^k - \frac{2}{3}} & \text{if } k \text{ is odd,} \\ d_1^{\frac{1}{3}2^k - \frac{2}{3}} d_2^{\frac{2}{3}2^k - \frac{1}{3}} & \text{if } k \text{ is even.} \end{cases}$$

However, one expects that the limits exist for “most” sequences γ.

To make this guess precise, we use the language (and tools) from probability.

To make this guess precise, we use the language (and tools) from probability.

- Fix a probability measure ν on S.
- Extend ν to a probability measure $\bar{\nu}$ on $\Phi_S = S^\infty = \Pi_{n=1}^\infty S$ via the product measure (i.i.d sequences).

To make this guess precise, we use the language (and tools) from probability.

- Fix a probability measure ν on S.
- Extend ν to a probability measure $\bar{\nu}$ on $\Phi_S = S^\infty = \Pi_{n=1}^\infty S$ via the product measure (i.i.d sequences).

Questions:

1. Do dynamical degrees exist almost surely?

To make this guess precise, we use the language (and tools) from probability.

- Fix a probability measure ν on S.
- Extend ν to a probability measure $\bar{\nu}$ on $\Phi_S = S^\infty = \Pi_{n=1}^\infty S$ via the product measure (i.i.d sequences).

Questions:

1. Do dynamical degrees exist almost surely?
2. It is known that when $S = \{\phi\}$, or for constant sequences, the dynamical degree bounds the arithmetic degree:

$$\boxed{\limsup_{n\to\infty} h(\phi^n(P))^{1/n} \leqslant \lim_{n\to\infty} \deg(\phi^n)^{1/n}}.$$

Is there such a statement for random sequences?

Yes, but we need a condition for rational maps:

Yes, but we need a condition for rational maps:

Definition: The set S is called *degree independent* if $\deg(f) \geqslant 2$ for all f in the semigroup generated by S.

Yes, but we need a condition for rational maps:

Definition: The set S is called *degree independent* if $\deg(f) \geqslant 2$ for all f in the semigroup generated by S.

Theorem (WH 2019)

Let S be a finite set of dominant rational self-maps on $\mathbb{P}^N(\overline{\mathbb{Q}})$ and let ν be a probability measure on S. Then:

Yes, but we need a condition for rational maps:

Definition: The set S is called *degree independent* if $\deg(f) \geqslant 2$ for all f in the semigroup generated by S.

Theorem (WH 2019)

Let S be a finite set of dominant rational self-maps on $\mathbb{P}^N(\overline{\mathbb{Q}})$ and let ν be a probability measure on S. Then:

(1) *There is a constant $\delta_{S,\nu}$ such that the limits*

$$\lim_{n\to\infty} \deg(\gamma_n^-)^{1/n} = \delta_{S,\nu} = \lim_{n\to\infty} \deg(\gamma_n^+)^{1/n}$$

hold (simultaneously) for $\bar{\nu}$-almost every $\gamma \in \Phi_S$.

Yes, but we need a condition for rational maps:

Definition: The set S is called *degree independent* if $\deg(f) \geqslant 2$ for all f in the semigroup generated by S.

Theorem (WH 2019)

Let S be a finite set of dominant rational self-maps on $\mathbb{P}^N(\overline{\mathbb{Q}})$ and let ν be a probability measure on S. Then:

(1) *There is a constant $\delta_{S,\nu}$ such that the limits*

$$\lim_{n\to\infty} \deg(\gamma_n^-)^{1/n} = \delta_{S,\nu} = \lim_{n\to\infty} \deg(\gamma_n^+)^{1/n}$$

hold (simultaneously) for $\bar{\nu}$-almost every $\gamma \in \Phi_S$.

(2) *If S is degree independent, then for $\bar{\nu}$-almost every $\gamma \in \Phi_S$ the bounds*

$$\limsup_{n\to\infty} h(\gamma_n^{\pm}(P))^{1/n} \leqslant \delta_{S,\nu}$$

hold (simultaneously) for all $P \in \mathbb{P}^N(\overline{\mathbb{Q}})_S$.

Remarks:

- The main tool is Kingman's Subadditive Ergodic Theorem (sort of strong law of large numbers for subadditive seq.).

Remarks:

- The main tool is Kingman's Subadditive Ergodic Theorem (sort of strong law of large numbers for subadditive seq.).

- If S is a set of **morphisms**, we can actually compute:

$$\delta_{S,\nu} = \prod_{\phi \in S} \deg(\phi)^{\nu(\phi)}.$$

Remarks:

- The main tool is Kingman's Subadditive Ergodic Theorem (sort of strong law of large numbers for subadditive seq.).
- If S is a set of **morphisms**, we can actually compute:

$$\delta_{S,\nu} = \prod_{\phi \in S} \deg(\phi)^{\nu(\phi)}.$$

There are infinite sets of morphisms where Theorem holds:

Remarks:

- The main tool is Kingman's Subadditive Ergodic Theorem (sort of strong law of large numbers for subadditive seq.).
- If S is a set of **morphisms**, we can actually compute:

$$\delta_{S,\nu} = \prod_{\phi \in S} \deg(\phi)^{\nu(\phi)}.$$

There are infinite sets of morphisms where Theorem holds:

$$|h \circ \phi - \deg(\phi) h| \leqslant C_{\phi},$$

Remarks:

- The main tool is Kingman's Subadditive Ergodic Theorem (sort of strong law of large numbers for subadditive seq.).

- If S is a set of **morphisms**, we can actually compute:

$$\delta_{S,\nu} = \prod_{\phi\in S} \deg(\phi)^{\nu(\phi)}.$$

There are infinite sets of morphisms where Theorem holds:

$$|h\circ\phi - \deg(\phi)h| \leqslant C_\phi,$$

need $\boxed{\sup_{\phi\in S}\{C_\phi\} < \infty}$ and $\boxed{\deg(\phi) \geqslant 2}$.

Remarks:

- The main tool is Kingman's Subadditive Ergodic Theorem (sort of strong law of large numbers for subadditive seq.).
- If S is a set of **morphisms**, we can actually compute:

$$\delta_{S,\nu} = \prod_{\phi \in S} \deg(\phi)^{\nu(\phi)}.$$

There are infinite sets of morphisms where Theorem holds:

$$|h \circ \phi - \deg(\phi) h| \leqslant C_\phi,$$

need $\boxed{\sup_{\phi \in S}\{C_\phi\} < \infty}$ and $\boxed{\deg(\phi) \geqslant 2}$.

Example: $S = \{x^d + c : \ d \geqslant 2, \ c \in \mathbb{Z}, \ |c| \leqslant B\}$.

Since the expected bound(s) hold,

$$\boxed{\limsup_{n\to\infty} h(\gamma_n^{\pm}(P))^{1/n} \leqslant \delta_{S,\nu}} \qquad \text{(almost surely)},$$

Since the expected bound(s) hold,

$$\boxed{\limsup_{n\to\infty} h(\gamma_n^{\pm}(P))^{1/n} \leqslant \delta_{S,\nu}} \qquad \text{(almost surely)},$$

one would like to know if/when:

- Replace $\leqslant$ with $=$
- Replace $\limsup\limits_{n\to\infty}$ with $\lim\limits_{n\to\infty}$

Since the expected bound(s) hold,

$$\boxed{\limsup_{n\to\infty} h(\gamma_n^{\pm}(P))^{1/n} \leqslant \delta_{S,\nu}} \qquad \text{(almost surely)},$$

one would like to know if/when:

- Replace $\leqslant$ with $=$
- Replace $\limsup_{n\to\infty}$ with $\lim_{n\to\infty}$

Note:
Some care must be taken (even for morphisms), since this would imply a sort of independence of the direction of iteration.

Example:

- Let $\phi_1 = x^2 - x$ and $\phi_2 = 3x^2$.
- Let ν on $S = \{\phi_1, \phi_2\}$ be given by $\nu(\phi_1) = 1/2 = \nu(\phi_2)$.
- Let $P = 1$.

Example:

- Let $\phi_1 = x^2 - x$ and $\phi_2 = 3x^2$.
- Let ν on $S = \{\phi_1, \phi_2\}$ be given by $\nu(\phi_1) = 1/2 = \nu(\phi_2)$.
- Let $P = 1$.

Then the growth rates below hold almost surely,

$$\liminf_{n\to\infty} h(\gamma_n^+(P))^{1/n} = 0 \quad \text{and} \quad \limsup_{n\to\infty} h(\gamma_n^+(P))^{1/n} = 2,$$

Example:

- Let $\phi_1 = x^2 - x$ and $\phi_2 = 3x^2$.
- Let ν on $S = \{\phi_1, \phi_2\}$ be given by $\nu(\phi_1) = 1/2 = \nu(\phi_2)$.
- Let $P = 1$.

Then the growth rates below hold almost surely,

$$\liminf_{n\to\infty} h(\gamma_n^+(P))^{1/n} = 0 \quad \text{and} \quad \limsup_{n\to\infty} h(\gamma_n^+(P))^{1/n} = 2,$$

and the growth rates below hold with probability 1/2:

$$\liminf_{n\to\infty} h(\gamma_n^-(P))^{1/n} = 0 \quad \text{and} \quad \limsup_{n\to\infty} h(\gamma_n^-(P))^{1/n} = 0$$

$$\liminf_{n\to\infty} h(\gamma_n^-(P))^{1/n} = 2 \quad \text{and} \quad \limsup_{n\to\infty} h(\gamma_n^-(P))^{1/n} = 2$$

Example:

- Let $\phi_1 = x^2 - x$ and $\phi_2 = 3x^2$.
- Let ν on $S = \{\phi_1, \phi_2\}$ be given by $\nu(\phi_1) = 1/2 = \nu(\phi_2)$.
- Let $P = 1$.

Then the growth rates below hold almost surely,

$$\liminf_{n\to\infty} h(\gamma_n^+(P))^{1/n} = 0 \quad \text{and} \quad \limsup_{n\to\infty} h(\gamma_n^+(P))^{1/n} = 2,$$

and the growth rates below hold with probability 1/2:

$$\liminf_{n\to\infty} h(\gamma_n^-(P))^{1/n} = 0 \quad \text{and} \quad \limsup_{n\to\infty} h(\gamma_n^-(P))^{1/n} = 0$$

$$\liminf_{n\to\infty} h(\gamma_n^-(P))^{1/n} = 2 \quad \text{and} \quad \limsup_{n\to\infty} h(\gamma_n^-(P))^{1/n} = 2$$

The direction of iteration may affect heights.

However, we can improve the upper bound to an equality and the limsup to a limit for points of sufficiently large height.

However, we can improve the upper bound to an equality and the limsup to a limit for points of sufficiently large height.

Theorem (WH 2019)

Let S be a finite set of endomorphisms of $\mathbb{P}^N(\overline{\mathbb{Q}})$ all of degree at least 2. Then there exists a constant B_S such that the following statements hold:

(1) *The dynamical degree is given by* $\delta_{S,\nu} = \prod_{\phi\in S} \deg(\phi)^{\nu(\phi)}$.

(2) *For $\bar{\nu}$-almost every $\gamma \in \Phi_S$, the limits*

$$\lim_{n\to\infty} h(\gamma_n^-(P))^{1/n} = \delta_{S,\nu} = \lim_{n\to\infty} h(\gamma_n^+(P))^{1/n}$$

hold (simultaneously) for all P with $h(P) > B_S$.

Application: height counting in orbits.

Height growth rates are independent of direction, "generically".

Application: height counting in orbits.

Height growth rates are independent of direction, "generically".

Corollary (WH 2019)

Let S be a finite set of endomorphisms all of degree at least 2. Then, outside of a set of points P of bounded height,

$$\lim_{B\to\infty} \frac{\#\{Q \in \mathrm{Orb}_\gamma^-(P) \;:\; h(Q) \leqslant B\}}{\log(B)} =$$

$$= \frac{1}{\log(\delta_{S,\nu})}$$

$$\lim_{B\to\infty} \frac{\#\{W \in \mathrm{Orb}_\gamma^+(P) \;:\; h(W) \leqslant B\}}{\log(B)} =$$

hold $\bar{\nu}$-almost surely.

Open problems

Questions:

1. Given S, is there a reasonable way to ensure that S is degree independent?
2. Amerik type result: is $\mathbb{P}^N(\overline{\mathbb{Q}})_S$ Zariski dense in $\mathbb{P}^N(\overline{\mathbb{Q}})$?
3. Given a finite set of monomial maps, can you compute $\delta_{S,\nu}$? (Random matrix theory problem)
4. Suppose $V \neq \mathbb{P}^N$. Can you prove $\delta_{S,\nu}$ exists and prove

$$\limsup_{n\to\infty} h(\gamma_n^{\pm}(P)) \leqslant \delta_{S,\nu}$$

almost surely (like we did for $\mathbb{P}^N$)?

Back to total orbits (for morphisms)

Given S and P, we can ask about the growth rate of

$$\#\{Q \in \mathrm{Orb}_S(P)\ :\ h(Q) \leqslant B\}.$$

Intuitively, this should (at least for generic P) depend on the Monoid M_S, i.e., the relations between the maps in S.

Back to total orbits (for morphisms)

Given S and P, we can ask about the growth rate of

$$\#\{Q \in \mathrm{Orb}_S(P) \;:\; h(Q) \leqslant B\}.$$

Intuitively, this should (at least for generic P) depend on the Monoid M_S, i.e., the relations between the maps in S.

Idea: Since, $h(f(P))$ behaves like $\deg(f)$, we are in some sense counting the number of f's in M_S of bounded degree.

Back to total orbits (for morphisms)

Given S and P, we can ask about the growth rate of

$$\#\{Q \in \mathrm{Orb}_S(P) \;:\; h(Q) \leqslant B\}.$$

Intuitively, this should (at least for generic P) depend on the Monoid M_S, i.e., the relations between the maps in S.

Idea: Since, $h(f(P))$ behaves like $\deg(f)$, we are in some sense counting the number of f's in M_S of bounded degree.

Formally, $\log\deg(f)$ defines a "length" on M_S, and the orbit count above is related to a growth rate (of "lengths") on M_S.

Some work on this type of problem has been done by group theorists.

Example:

$$\boxed{\lim_{B\to\infty} \frac{\#\Big\{f \in M_S \;:\; h\big(f(P)\big) \leqslant B\Big\}}{(\log B)^s} = \frac{1}{s!\cdot \prod_{i=1}^{s} \log \deg(\phi_i)},}$$

when S is a free basis for the commutative monoid M_S and P has sufficiently large height.

Remark: Work is ongoing to compute this growth rate for free (non-commutative) monoids. Others?

Thank you!!

Questions or comments? Please send them to:

wmh33@txstate.edu