Post-critically finite cubic polynomials

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Definitions

- Let K be a number field.
- $f \in K(x)$ is a rational function (morphism of $\mathbb{P}^1$).
- Conjugacy class: $[f] = \{\phi \circ f \circ \phi^{-1} : \phi \in \mathrm{PGL}_2(\bar{K})\}$.
- $\mathrm{Crit}(f) = \{\text{critical points of } f\} = \{\alpha \in \mathbb{P}^1 : f'(\alpha) = 0\}$.
- Orbit of a point $\alpha \in \mathbb{P}^1 = \{f^n(\alpha) : n \geq 0\}$.

Definition

f is post-critically finite (PCF) if every element of $\mathrm{Crit}(f)$ has finite forward orbit.

Ingram, 2011

Theorem

The set of conjugacy classes of post-critically finite polynomials of degree d with coefficients of algebraic degree at most B is a finite and effectively computable set.

Application

If $f(z) = z^3 + Az + B$ has coefficients in $\mathbb{Q}$ and is post-critically finite, then

$$(A, B) \in \left\{(-3, 0), \left(-\frac{3}{2}, 0\right), \left(-\frac{3}{4}, \frac{3}{4}\right), \left(-\frac{3}{4}, -\frac{3}{4}\right), (0, 0), \left(\frac{3}{2}, 0\right), (3, 0)\right\}.$$

Ingram, 2011

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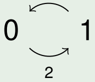
Missing some cubic PCF polynomials

$f(z) = -z^3 + 1$



Problem: Monic centered form $f(z) = z^3 + Az + B$ does not preserve field of definition.

Motivation

Question

Can we use Ingram's techniques and a different normal form to find **all** PCF cubic polynomials defined over $\mathbb{Q}$ (up to conjugacy over $\bar{\mathbb{Q}}$)?

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Can we use Ingram's techniques and a different normal form to find **all** PCF cubic polynomials defined over $\mathbb{Q}$ (up to conjugacy over $\bar{\mathbb{Q}}$)?

Spoiler: Yes.

Strategy

1. Find normal forms that respect the field of definition.
2. For a map to be PCF, it must be post-critically bounded in each absolute value. Find archimedean and p-adic bounds on the coefficients for maps in the normal forms to be post-critically bounded.
3. Use the bounds in 2 to create a finite search space of possibly PCF maps.
4. For each map in the finite search space, test if it is PCF or not.

All cubic PCF polynomials

Theorem

There are exactly fifteen $\bar{\mathbb{Q}}$ conjugacy classes of cubic PCF polynomials defined over $\mathbb{Q}$:

$$
\begin{array}{llllll}
(1) & z^3 & (2) & -z^3+1 & (3) & -2z^3+3z^2+\frac{1}{2} \\
(4) & -2z^3+3z^2 & (5) & -z^3+\frac{3}{2}z^2-1 & (6) & 2z^3-3z^2+1 \\
(7) & 2z^3-3z^2+\frac{1}{2} & (8) & z^3-\frac{3}{2}z^2 & (9) & -3z^3+\frac{9}{2}z^2 \\
(10) & -4z^3+6z^2-\frac{1}{2} & (11) & 4z^3-6z^2+\frac{3}{2} & (12) & 3z^3-\frac{9}{2}z^2+1 \\
(13) & -z^3+\frac{3}{2}z^2-1 & (14) & -\frac{1}{4}z^3+\frac{3}{2}z+2 & (15) & -\frac{1}{28}z^3-\frac{3}{4}z+\frac{7}{2}
\end{array}
$$

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(4)	$-2z^3+3z^2$	(5)	$-z^3+\frac{3}{2}z^2-1$	(6)	$2z^3-3z^2+1$
(7)	$2z^3-3z^2+\frac{1}{2}$	(8)	$z^3-\frac{3}{2}z^2$	(9)	$-3z^3+\frac{9}{2}z^2$
(10)	$-4z^3+6z^2-\frac{1}{2}$	(11)	$4z^3-6z^2+\frac{3}{2}$	(12)	$3z^3-\frac{9}{2}z^2+1$
(13)	$-z^3+\frac{3}{2}z^2-1$	(14)	$-\frac{1}{4}z^3+\frac{3}{2}z+2$	(15)	$-\frac{1}{28}z^3-\frac{3}{4}z+\frac{7}{2}$

Strategy

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Setup

Let K be a number field, and let $f(z) \in K[z]$ be a cubic polynomial. Possibilities:

1. There is exactly one critical point, $\gamma \in K$.
2. There are two distinct critical points: $\gamma_1 \neq \gamma_2$, and they are both K-rational.
3. There are two distinct critical points $\gamma_1 \neq \gamma_2$ with $K(\gamma_1) = K(\gamma_2)$ a quadratic extension of K.

Case 1 f is **unicritical**.
Cases 2 and 3 : f is **bicritical**.

Unicritical polynomials

Theorem

Let $f(z) \in K[z]$ be a degree d unicritical polynomial. Then either $f(z)$ is $\bar{K}$-conjugate to z^d, or f is conjugate to a unique polynomial of the form

$$az^d + 1 \in K[z].$$

Rational critical points

Dynamical Belyi polynomials

Degree d. Fixed critical points at 0 and 1.
$d - k =$ ramification index of 0.

$$\mathcal{B}_{d,k}(z) = \left(\frac{1}{k!} \prod_{j=0}^{k} (d-j) \right) x^{d-k} \sum_{i=1}^{k} \frac{(-1)^i}{(d-k+i)} \binom{k}{i} x^i.$$

Proposition

Let $g \in K[z]$ be a bicritical polynomial of degree $d \geq 3$ with $\mathrm{Crit}(g) = \{\gamma_1, \gamma_2\} \subseteq K$. There exists an element $\phi \in \mathrm{PGL}_2(K)$ such that $g^\phi = a\mathcal{B}_{d,k} + c$ for some $k \in \mathbb{N}$ and some $a, c \in K$.

Bicritical polynomials

Field of definition preserved

Let

$$f(z) = \frac{z^3}{4} - \frac{3z}{2}, \text{ so } \operatorname{Crit}(f) = \left\{\pm\sqrt{2}\right\}.$$

Moving the two critical points to 0 and 1 gives the polynomial

$$g(z) = 2z^3 - 3z^2 + 1.$$

Both polynomials — one with rational critical points and one with irrational critical points — are defined over $\mathbb{Q}$.

Bicritical polynomials

Field of definition not preserved

Let

$$f(z) = -\frac{z^3}{4} + \frac{3z}{2} + 2, \text{ so } \operatorname{Crit}(f) = \left\{\pm\sqrt{2}\right\}.$$

Moving the two critical points to 0 and 1 gives the polynomial

$$g(z) = -2z^3 + 3z^2 - \frac{1}{\sqrt{2}}.$$

Irrational critical points

Irrational critical points

Degree d. Fixed point at 0. Critical points at $\pm\sqrt{D}$.

$$\mathcal{P}_{d,D}(z) = \sum_{j=0}^{\frac{d-1}{2}} (-D)^{\frac{d-1}{2}-j} \binom{\frac{d-1}{2}}{j} \frac{z^{2j+1}}{2j+1}.$$

Proposition

Let $g(z) \in K[z]$ be a bicritical polynomial of degree $d \geq 3$. Suppose that $\mathrm{Crit}(g) = \{\gamma_1, \gamma_2\} \not\subset K$. Then g is conjugate to a map of the form $a\mathcal{P}_{d,D}(z) + c$ for some $a, c \in K$ and some $D \in \mathcal{O}_K^\times / \mathcal{O}_K^2$.

Strategy

1. Find normal forms that respect the field of definition.
2. For a map to be PCF, it must be post-critically bounded in each absolute value. Find archimedean and p-adic bounds on the coefficients for maps in the normal forms to be post-critically bounded.
3. Use the bounds in 2 to create a finite search space of possibly PCF maps.
4. For each map in the finite search space, test if it is PCF or not.

Notation

From Ingram:

$$f(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0 \in K[z]$$

$$(2d)_\nu = \begin{cases} 1 & \nu \text{ is non-archimedean} \\ 2d & \nu \text{ is archimedean} \end{cases}$$

$$C_{f,\nu} = (2d)_\nu \max_{0 \leq i < d} \left\{ 1, \left| \frac{a_i}{a_d} \right|_\nu^{\frac{1}{d-i}}, |a_d|_\nu^{-\frac{1}{d-1}} \right\}$$

Lemma

Let $f(z) \in \mathbb{Q}[z]$ be a polynomial of degree $d \geq 2$. For $\alpha \in \mathbb{Q}$, if there exists $\nu \in M_{\mathbb{Q}}$ and $n \in \mathbb{N}$ such that $\left| f^n(\alpha) \right|_\nu > C_{f,\nu}$, then α must be a wandering point for f.

Unicritical polynomials

Theorem

Let $f(z) = az^d + 1 \in \mathbb{Q}[z]$ and $d \geq 2$. For d even, f is PCF if and only if $a \in \{-2, -1\}$. For d odd, f is PCF if and only if $a = -1$.

Proof.

$C_{f,p} = \left\{1, |a|_p^{-1/(d-1)}\right\}$.
Note $f^2(0) = a + 1$, so require

$$|a+1|_p \leq \max\{|a|_p, 1\} \leq C_{f,p} \text{ for all primes } p.$$

Get $|a|_p \leq 1$ for all primes p. Also $|a| \leq 2$.
Check $a = \pm 1, \pm 2$. □

Rational critical points

Let $f(z) = a(-2z^3 + 3z^2) + c \in \mathbb{Q}[z]$. If f is PCF,

$$\left|f(1)\right| = |a + c|_\nu \leq C_{f,\nu} \quad \text{and} \quad \left|f(0)\right| = |c|_\nu \leq C_{f,\nu}.$$

For non-archimedean place ν, $\max\{|a|_\nu, |c|_\nu\} \leq C_{f,\nu}$.

Rational critical points

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For non-archimedean place ν, $\max\{|a|_\nu, |c|_\nu\} \le C_{f,\nu}$.

Proposition

If $f_{a,c}(z) = a(-2z^3 + 3z^2) + c \in \mathbb{Q}[z]$ is PCF, then

$$\pm a \in \left\{\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}\right\} \text{ and } \pm c \in \left\{0, 1, \frac{1}{2}, \frac{3}{2}, 2\right\}.$$

Rational critical points

Theorem

If $f(z) \in \mathbb{Q}[z]$ is a cubic bicritical PCF polynomial with rational critical points, then $f(z)$ is conjugate to $f_{a,c}(z) = a(-2z^3 + 3z^2) + c$ where

$$(a, c) \in \left\{(1, 0), \left(\pm 1, \frac{1}{2}\right), \left(\frac{1}{2}, \pm 1\right), \left(2, -\frac{1}{2}\right), \left(\frac{3}{2}, 0\right), (-1, 1), \left(-2, \frac{3}{2}\right), \left(-\frac{3}{2}, 1\right), \left(-\frac{1}{2}, 0\right)\right\}.$$

Proof.

We have 126 possibilities for (a, c). Test if they are PCF or not using Sage. □

Irrational critical points

Lemma

Let $f(z) = a(z^3/3 - Dz) + c \in \mathbb{Q}[z]$*. If* f *is PCF, then*

$$\pm aD \in \left\{ \frac{3}{4}, \frac{3}{2}, \frac{9}{4}, 3, \frac{15}{4}, \frac{9}{2}, \frac{21}{4} \right\}.$$

Lemma

Let $f(z) = a(z^3/3 - Dz) + c \in \mathbb{Q}[z]$*. If* f *is PCB in the archimedean place, then* $|c|^2 < 11|D|$*.*

Lemma

Let $f(z) = a(z^3/3 - Dz) + c \in \mathbb{Q}[z]$*. If* f *is p-adically PCB, then*

$$\left| c\sqrt{a} \right|_p \leq \begin{cases} 1 & \text{if } p \geq 5 \\ 3^{-1/2} & \text{if } p = 3 \\ 2^3 & \text{if } p = 2. \end{cases}$$

Irrational critical points

Theorem

If $f(z) \in \mathbb{Q}[z]$ is a cubic bicritical PCF polynomial that is not conjugate to a polynomial with rational critical points, then $f(z)$ is conjugate to $f_{D,a,c}(z) = a(\frac{z^3}{3} - Dz) + c$ *where*

$$(D, a, c) \in \left\{ \left(2, -\frac{3}{4}, 2\right), \left(-7, -\frac{3}{28}, \frac{7}{2}\right) \right\}.$$

Algorithm ($D \in \mathbb{Z}$, odd squarefree)

Step 1 **Loop over possible aD values.**

$$\pm aD \in \left\{ \frac{3}{4}, \frac{3}{2}, \frac{9}{4}, 3, \frac{15}{4}, \frac{9}{2}, \frac{21}{4} \right\}.$$

Step 2 **Compute $|a|_2$.**

Step 3 **Find an upper bound for $|c|_p$ for each prime p.**

$$\left|c\sqrt{a}\right|_p \leq \begin{cases} 1 & \text{if } p \geq 5 \\ 3^{-1/2} & \text{if } p = 3 \\ 2^3 & \text{if } p = 2. \end{cases}$$

So we can find $e \leq 3$ such that $|c|_2 \leq 2^e$, and $|c|_p \leq 1$ for each prime $p \geq 3$.

Algorithm ($D \in \mathbb{Z}$, odd squarefree)

Step 4 **Factor D and c.** Write $D = mP$, where m and P are relatively prime odd squarefree integers, m divides numerator of aD and P divides denominator of a. Then P must also divide the numerator of c, so $c = \frac{Pk}{2^e}$ for some positive integer k.

Step 5 **Bound the factors of D and c.** Use $|c|^2 < 11|D|$, so $\frac{P^2k^2}{2^{2e}} < 11mP$. Therefore $Pk^2 < B$ where $B = 11m \cdot 2^{2e}$.

Step 6 **Loop over P values.** For all odd, squarefree integers $P < B$, determine the set of possible k values such that $Pk^2 < B$.

Algorithm ($D \in \mathbb{Z}$, odd squarefree)

Step 7 **Create the triple.** Each triple (m, P, k) yields a triple

$$(D, a, c) = \left(mP, \frac{aD}{mP}, \frac{Pk}{2^e} \right).$$

Finally, check that $3|ac$ to verify that the triple satisfies the 3-adic condition. If so, add (D, a, c) to the list of possible PCF triples.

Proof.

These algorithms yield a list of 5,957 triples corresponding to 23,828 possibly PCF polynomials.

Only the two listed in the theorem statement are actually PCF and are not conjugate to a polynomial already with rational critical points. □

Potential good reduction

Let K be a number field, let $f(z) \in K(z)$ be a rational function of degree $d \geq 2$.

Definition

We say f has **good reduction** *at a prime $\mathfrak{p}$ if $\deg \tilde{f} = \deg f$. We say f has* **potential good reduction** *at $\mathfrak{p}$ if it is $\bar{K}$-conjugate to a map with good reduction at $\mathfrak{p}$.*

If f does not have potential good reduction at $\mathfrak{p}$, we say it has **persistent bad reduction at** $\mathfrak{p}$.

Wishful thinking

PCF functions $\leftrightsquigarrow$ CM abelian varieties

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PCF functions $\leftrightsquigarrow$ $\underbrace{\text{CM abelian varieties}}_{\text{everywhere potential good reduction}}$

Wishful thinking

$$\underbrace{\text{PCF functions}}_{???} \quad \leftrightsquigarrow \quad \underbrace{\text{CM abelian varieties}}_{\text{everywhere potential good reduction}}$$

Bad news

The maps

$$f(z) = \frac{-4}{9z^2 - 12z} \quad \text{and} \quad g(z) = \frac{3z^2 - 4z + 1}{1 - 4z}$$

have persistent bad reduction at 3.

Polynomials

Theorem

If $f(z) \in \bar{\mathbb{Q}}[z]$ is PCF with degree $d \geq 2$ and $S_d = \{p \text{ prime} : p \leq d\}$, then f has potential good reduction outside S_d.

Proof.

Let $p > d$, and $\sigma_i = \sigma_i(\text{crit pts})$. Conjugate so that

$$f(z) = z^d - \frac{d}{d-1}\sigma_1 z^{d-1} + \frac{d}{d-2}\sigma_2 z^{d-2} - \cdots + (-1)^{d-1} d\sigma_{d-1} z.$$

Then f is p-adically PCB iff $|\text{crit pts}|_p \leq 1$.

If f is PCF, each coefficient has p-adic absolute value ≤ 1. □

Can we say more?

- For quadratic polynomials, conjugate to $f(z) = z^2 + c$ preserves field of definition. If $f^m(0) = f^n(0)$, then c is a root of a monic polynomial with coefficients in $\mathbb{Z}$, so good reduction.
- Known (to me) examples of PCF maps with persistent bad reduction are rational maps.

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- For quadratic polynomials, conjugate to $f(z) = z^2 + c$ preserves field of definition. If $f^m(0) = f^n(0)$, then c is a root of a monic polynomial with coefficients in $\mathbb{Z}$, so good reduction.
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Question

Let $d \geq 3, p \leq d$. Can we find a PCF $f \in \mathbb{Q}[z]$ of degree d such that f has persistent bad reduction at p?

Partial answer

Proposition

Let $d \geq 3$. If $p|(d-1)$, then there exists a PCF polynomial $f(z) \in \mathbb{Q}[z]$ of degree d with persistent bad reduction at p. Namely,

$$f(z) = -B_{d,1}(z) + 1,$$

where $B_{d,1}$ is the dynamical Belyi map.

Proof.

Newton polygon to show f has a p-adically repelling fixed point. □

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Proof.

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Can extend this to $p > k$ and $p|(d-k)$, using the polynomial $-B_{d,k}(z) + 1$, for $1 \leq k < d-1$.

Thank you!