Higher Arithmetic Degrees

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Set-up

- X projective over number field K
- $f : X \dashrightarrow X$ dominant rational map
- How to measure the "dynamical complexity" of f?

First dynamical degree

- Measures growth rate of degrees of f^n:

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- (Theorem: Limit exists.)

Height functions

- Suppose $P \in \mathbb{P}^n(\mathbb{Q})$.
- Write $P = [X_0, \ldots, X_n] \in \mathbb{P}^n(\mathbb{Q})$ with coprime integers.
- Set $h(P) = \log(\max |X_i|)$.
- (Similar for number fields.)

Arithmetic degree

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- (Question: Limit exists?)

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- Define $f : \mathbb{A}^2 \to \mathbb{A}^2$ by $(x, y) \mapsto (y, xy)$.

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- f^n given by $(x^{F_n}y^{F_{n+1}}, x^{F_{n+1}}y^{F_{n+2}})$.

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- $\alpha_P(f) = 1$ if $P = (1, 1)$

The Kawaguchi–Silverman Conjecture

- Conj: If P has dense orbit, then limit defining $\alpha_f(P)$ exists and is equal to $\lambda_1(f)$.

Higher dynamical degrees

- Set

$$\lambda_k(f) = \lim_{n\to\infty} \left((f^n)^* H^k \cdot H^{\dim X - k}\right)^{1/n}$$

- Measures degree growth along orbit of a codimension-k cycle.

Properties of dynamical degrees

Limit exists:	Use Fekete's lemma
Log concavity:	$\lambda_{k-1}(f)\lambda_{k+1}(f) \leq \lambda_k(f)^2$
Inverses:	$\lambda_k(f) = \lambda_{\dim X-k}(f^{-1})$
Product formula:	$\lambda_k(f) = \max(\lambda_i(\text{base})\lambda_{k-i}(\text{fibers}))$
Biratl surface maps:	Salem or Pisot

Goal: Arithmetic analogs

- Want to define higher arithmetic degrees measuring height growth of higher-dimensional cycles.
- Problem: What could log concavity even say in this case?
- Each of the numbers involved depends on a choice of cycle!

Plan

- Define $\alpha_k(f)$, independent of cycle, which we imagine as "generic" height growth.
- Use an intersection in Arakelov Chow formally identical to the one appearing in dynamicall degrees.
- Many proofs become a copy/paste, replacing inequalities on intersections with corresponding arithmetic results!

Definition

- If V has dimension k, set

$$\alpha_{k+1}(f;V) = \limsup_{n\to\infty} \left(\widehat{\deg}\left((f^n)^* H \cdot V \right) \right)^{1/n}$$

- Set

$$\alpha_k(f) = \limsup_{n\to\infty} \left(\widehat{\deg}\left((f^n)^* H^k \cdot H^{\dim X + 1 - k} \right) \right)^{1/n}$$

- Note: growth of a 1-cycle is $\alpha_2(f;V)$; awkward indexing.

KS conjecture: heuristic

- Consider the function field analog:
- X defined over $K(C)$
- Pick a model $\pi : \mathcal{X} \to C$
- $h(V) = (\mathcal{V} \cdot H)_{\mathcal{X}}$
- How do we expect $h(f^n(V))$ to grow?

KS conjecture: heuristic

- V dimension $k+1$ in X, so $\mathcal{V}$ dimension $k+2$ in $\mathcal{X}$

$$\alpha_k(f;V) = \lambda_k(f_{\mathcal{X}}) = \max(\lambda_k(f), \lambda_{k-1}(f))$$

by the product formula.

Higher codimension KS

- We split the conjecture into two parts:
- I: $\alpha_k(f) = \max(\lambda_k(f), \lambda_{k-1}(f))$
- II: $\alpha_k(f; V) = \alpha_k(f)$ for "general" V (dense orbit?)
- Expected height growth:

$$\alpha_k(f; V) = \max(\lambda_k(f), \lambda_{k-1}(f))$$

On a surface

- Growth rate of a curve should be

$$\alpha_2(f) = \max(\lambda_1(f), \lambda_2(f)).$$

- Both can occur!
- If X is birational can have $\lambda_1 > \lambda_2 = 1$
- If X monomial can have $\lambda_2 > \lambda_1$.

What we know

- Thm: Limit defining α_1 exists and is equal to λ_1.
- (Limit for α_k will be OK pending arithmetic version of an inequality of Siu.)
- Thm: The limsup defining $\alpha_k(f)$ is finite (bounded by $\lambda_1(f)^k$

What we know

- Thm: Log concavity holds (use arithmetic Teissier–Khovanskii inequalities of Ikoma, Yuan–Zhang)
- Thm: Inverse property for birational maps holds (it's just push/pull)

KS, part I

- Conjecture: $\alpha_{k+1}(f) = \max(\lambda_k(f), \lambda_{k+1}(f))$.
- Theorem: $\lambda_1(f) = \alpha_1(f)$.
- Theorem: $\alpha_{k+1}(f) \geq \max(\lambda_k(f), \lambda_{k+1}(f))$