New types of heights with connections to the Batyrev–Manin and Malle Conjectures

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(joint with Jordan Ellenberg and David Zureick-Brown)

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Counting Rational Points

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- When $X = pt$, $\mathcal{X} = BG$, we recover disc

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- Need heights of vector bundles otherwise can't distinguish between BG and BG^{ab}.

Failure of the valuative criterion

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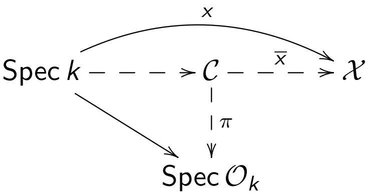
Definition of stacky heights



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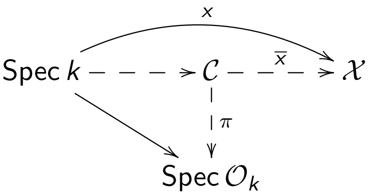

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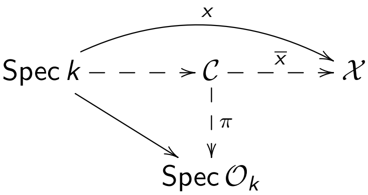

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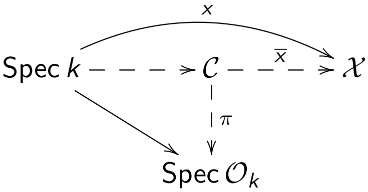

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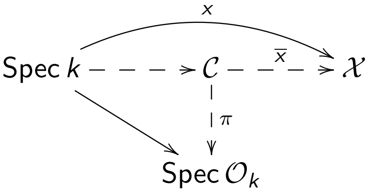

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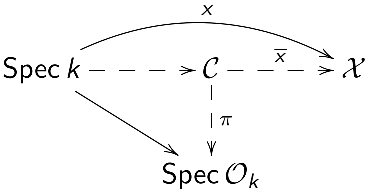

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- $\mathcal{V} = \text{regular rep: } h_{\mathcal{V}}(x) = \frac{1}{2} \log |N_{k/\mathbb{Q}} \operatorname{disc}(L/k)|$
- For Malle, use $\mathcal{V} = \text{permutation rep of } G \subseteq S_n$.

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Definition

$\mathrm{edd}(x) := \deg(\overline{x}^* T_{\mathcal{X}}) - \deg(T_{\mathcal{C}})$, where $\overline{x}$ is representable.

Stacky Batyrev–Manin–Malle Conjecture

Definition

$\mathcal{V}$ is *generically semipositive* if there is a closed substack $\mathcal{Z} \subsetneq \mathcal{X}$ such that for all B and every extension finite L/k,

$$\{x \in \mathcal{X}(L) : h_{\mathcal{V}}(x) < B\}$$

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Definition

A function $f : \mathcal{X}(\overline{k}) \to \mathbb{R}$ is *generically bounded below* if for all $d \in \mathbb{Z}^+$, there is a constant B_d such that

$$\bigcup_{[L:k]=d} \{x \in \mathcal{X}(L) : h_{\mathcal{V}}(x) < B_d\}$$

is not Zariski dense.

Stacky Batyrev–Manin–Malle Conjecture

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Let $\mathcal{X}$ *be a smooth proper stack such that* edd *is generically bounded below. If* $\mathcal{V}$ *is a generically semipositive vector bundle, then there exists* $\mathcal{U} \subseteq \mathcal{X}$ *open dense such that for all* $\epsilon > 0$*,*

$$N_{\mathcal{U},\mathcal{V},k}(B) = O(B^{a(\mathcal{V})+\epsilon})$$

where $a(\mathcal{V}) = \inf\{t \mid th_{\mathcal{V}} - \mathrm{edd}\text{ generically bounded below}\}$.

Thank you.