Automorphism loci for endomorphisms of $\mathbb{P}^1$

Benjamin Hutz

Department of Mathematics and Statistics
Saint Louis University

January 16–19, 2020
JMM Denver, CO
Special Session on Arithmetic Dynamics

Collaborators!

Summer@ICERM 2019 - Computational Arithmetic Dynamics

Char 0

- Brandon Gontmacher - Stony Brook University
- Grayson Jorgenson - Florida State University
- Srinjoy Srimani - Brown University
- Simon Xu - Colby College

Char $p > 0$

- Julia Cai - Yale University
- Leo Mayer - Lawrence University
- Max Weinreich - Brown University

Thanks to ICERM for supporting the program and the other faculty mentors and research assistants who made it possible!

The Main Questions

Question 1

Describe the locus $\mathcal{A}_d \subset \mathcal{M}_d$ of conjugacy classes of morphisms of degree d which have a non-trivial automorphism group.

Question 2

Which finite subgroups $\Gamma \subseteq \mathrm{PGL}_2$ occur as the automorphism group of some $f : \mathbb{P}^1 \to \mathbb{P}^1$?

Moduli Space of Dynamical Systems

Definition

Define Hom_d to be the set of degree d endomorphisms of the projective line.

Definition

Given $f \in \mathrm{Hom}_d$ we can conjugate by an element $\alpha \in \mathrm{PGL}_2 = \mathrm{Aut}(\mathbb{P}^1)$

$$f^\alpha = \alpha^{-1} \circ f \circ \alpha.$$

This preserves the dynamical properties of f: $(f^n)^\alpha = (f^\alpha)^n$.

Definition

We define the moduli space of degree d morphisms as the quotient $\mathcal{M}_d = \mathrm{Hom}_d \,/\, \mathrm{PGL}_2$.

Automorphisms

Definition

For $f : \mathbb{P}^1 \to \mathbb{P}^1$ define the automorphism group of f as

$$\mathrm{Aut}(f) = \{\alpha \in \mathrm{PGL}_2 \mid f^\alpha = f\}.$$

Definition

Given the set of n-periodic points $\{P_1, \ldots, P_{n_d}\}$ and their multipliers $\Lambda_n = \{\lambda_1, \ldots, \lambda_{n_d}\}$.

Theorem (Milnor (1993), Silverman (1998))

The elementary symmetric polynomials evaluated on Λ_n are invariants of M_d.

Automorphisms

Theorem (Milnor (1993), Silverman (1998))

There is an isomorphism $\mathcal{M}_2 \cong \mathbb{A}^2$ *given by* $[f] \mapsto (\sigma_1, \sigma_2)$*.*

Theorem (Milnor (1993), Fujimura-Nishizawa (2007))

$\mathcal{A}_2(\overline{\mathbb{Q}})$ *is the cuspidal cubic*

$$2\sigma_1^3 + \sigma_1^2\sigma_2 - \sigma_1^2 - 4\sigma_2^2 - 8\sigma_1\sigma_2 + 12\sigma_1 + 12\sigma_2 - 36 = 0.$$

Theorem (CHMW)

$\mathcal{A}_2(\overline{\mathbb{F}}_2)$ *is a line missing a point given by* $\{\sigma_1 = 0 : \sigma_2 \neq 1\}$*. In particular,* $\mathcal{A}_2(\overline{\mathbb{F}_2})$ *is not Zariski closed.*

Degree 3 and 4

Related

- West (2014) - Parametrization of $\mathcal{M}_3$ from which the maps with non-trivial automorphisms could be extracted.

Tools

- Doyle-McMullen (1989) - invariant theory and automorphisms
- Miasnikov-Stout-Williams (2014) - Dimensions of $\mathcal{A}_d(\Gamma)$
- de Faria-Hutz (2015) - invariant theory and automorphisms

$\mathcal{A}_3$ - Miasnikov-Stout-Williams

Group	Dimension	Family
A_4	0	
C_4	0	
D_4	0	
C_3	1	
D_2	1	
C_2	2	

$\mathcal{A}_3$ - GHJSX

Group	Dimension	Family
A_4	0	to be revealed
C_4	0	$\frac{1}{z^3}$
D_4	0	$\frac{1}{z^3}$
C_3	1	$\frac{z^3+a}{az^2}$
D_2	1	$\frac{az^2-1}{z^3-az}$
	1	$\frac{az^2+1}{z^3+az}$
C_2	2	$\frac{az^2+1}{z^3+bz}$
	2	$\frac{z^3+az}{bz^2+1}$

$\mathcal{A}_4$ - Miasnikov-Stout-Williams

Group	Dimension	Family
D_5	0	
C_5	0	
C_4	1	
D_3	1	
C_3	2	
C_2	3	

$\mathcal{A}_4$ - GHJSX

Group	Dimension	Family
D_5	0	$\frac{1}{z^4}$
C_5	0	$\frac{1}{z^4}$
C_4	1	$\frac{z^4+1}{az^3}$
D_3	1	$\frac{z^4+az}{az^3+1}$
C_3	2	$\frac{z^4+az}{bz^3+1}$
C_2	3	$\frac{z^4+az^2+1}{bz^3+cz}$

Application to Uniform-Boundedness

Theorem (GHJSX)

Assuming a specific upper bound on the period of a $\mathbb{Q}$-rational periodic point for each family, then

$$\#\operatorname{PrePer}(f) \leq 8, \quad \forall f \in \mathcal{A}_3(\mathbb{Q})$$

with $\mathcal{A}_3$ represented as above.

A similar theorem for $\mathcal{A}_4$...

Existence of Automorphisms in Characteristic 0

Theorem (deFaria-H, 2015)

For any representation of a finite subgroup $\Gamma \subset \mathrm{PGL}_2$*, there are infinitely many maps whose automorphism group contains* Γ*.*

Proof.

The proof is constructive and relies on the invariant theory of finite groups. □

deFaria-H were also able to find a $\mathbb{Q}$-rational map for all finite subgroups of PGL_2 except for the tetrahedral group.

Field of Definition

Theorem (GHJSX)

Every finite subgroup of PGL_2 can be realized as the (exact) automorphism group of a map defined over $\mathbb{Q}$.

Proof.

The map

$$f(z) = \frac{z^3 - 3}{-3z^2}$$

has tetrahedral automorphism group. □

Sigma Invariants

Theorem (McMullen, 1987)

The multiplier map is finite-to-one away from the locus of Lattès maps.

Theorem (H.-Tepper, 2013)

The multiplier map on $\mathcal{M}_3$ is at most 12-to-1.

Open Problem

Exhibit an explicit 12-to-one example.

Interesting Family

The family of maps

$$f_a(z) = \frac{az^2 - 1}{z^3 - az}$$

has the following properties.

- $\mathrm{Aut}(f_a) \subset D_2$ for all a
- $\Lambda_1(f_a)$ does not depend on a

Characteristic $p > 0$

Complications

1. $\mathrm{PGL}_2(\mathbb{F}_q)$ is a finite group whereas $\mathrm{PGL}_2(\overline{\mathbb{F}}_p)$ is not. However, every element of $\mathrm{PGL}_2(\overline{\mathbb{F}}_p)$ lives in some $\mathrm{PGL}_2(\mathbb{F}_q)$, so we will consider each of these finite groups.
2. Invariant theory is more complicated when the characteristic divides the order of the group.
3. The constructive theorems in char 0 do not necessarily still work in char p.
4. There are additional conjugacy classes of representations.

Classification of finite subgroups

Theorem (Faber, 2012)

The conjugacy classes of finite subgroups of $\mathrm{PGL}_2(\overline{\mathbb{F}}_p)$ *are as follows.*

- *The p-regular case:* C_n, D_{2n}, A_4, A_5, S_4, *except when p divides the corresponding group order. Each occurs as just one conjugacy class.*
- *The p-irregular case:* $\mathrm{PGL}_2(\mathbb{F}_q)$, $\mathrm{PSL}_2(\mathbb{F}_q)$, *subgroups of* $B(\mathbb{F}_q)$. *The first two occur in just one conjugacy class, but the others sometimes occur with multiple conjugacy classes*

PGL($\mathbb{F}_q$)

Theorem (CHMW)

The map $f(z) = z^q$ has automorpism group $\mathrm{PGL}_2(\mathbb{F}_q)$. This is the unique conjuacy class of degree q which is rational over $\mathbb{F}_q$ with this automorphism group. This is also the minimal degree of such a map.

So the char 0 questions about the realizability of $\Gamma \subseteq \mathrm{Aut}(f)$ are now fully answered as "yes!" We look next at the more difficult question of (rational) realizability of exact automorphism groups.

One Example

Example

$\# \mathrm{PSL}_2(\mathbb{F}_q) = \frac{(q+1)q(q-1)}{2}$, $p > 2$, $\mathrm{PSL}_2(\mathbb{F}_q) \cong \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle$.

With an ad hoc brute force type approach the following were the (minimal) degrees of maps which had $\mathrm{PSL}_2(\mathbb{F}_q)$ as exact automorphism group

q	Degree
3	7
5	41
7	127
11	551
13	937

which seemed to be

$$\deg(f) = (q+1)\left(\frac{1}{2}q^2 - \frac{3}{2}q + 2\right) - 1$$

Better!

Theorem (CHMW)

Let u and c_1 be the fundamental invariants of $\mathrm{PSL}_2(\mathbb{F}_q)$ *of degree* $q+1$ *and* q^2-q*, respectively*

$$u = x^q y + xy^q$$

$$c_1 = \sum_{n=0}^{q} x^{(q-1)(q-n)} y^{(q-1)n}.$$

Define

$$f : \mathbb{P}^1 \to \mathbb{P}^1$$

$$f(x,y) = \left[-x\frac{c_1^b}{2} + \frac{\partial u^a}{\partial y}, y\frac{c_1^b}{2} - \frac{\partial u^a}{\partial x} \right],$$

where $b = \frac{q-1}{2}$ *and* $a = \frac{q(q-3)+4}{2}$*. Then* $\mathrm{Aut}(f) = \mathrm{PSL}_2(\mathbb{F}_q)$*.*

Some open questions

1. Properties of these families!
   1. Unconditional uniform boundedness for $\mathcal{A}_d$
   2. How automorphisms affect the "randomness" of the map
   3. Geometry of the families that have dimension > 1
2. Why are we seeing a symmetry disappear in the limit for $\mathcal{A}_2(\overline{\mathbb{F}}_p)$?
3. Description and geometry of $\mathcal{A}_3(\overline{\mathbb{F}}_p)$.
4. What is the appropriate reformulation for characteristic p of the theorem that $\mathcal{A}_d(\mathbb{C})$ is Zariski closed?
5. As the map f_c varies, so does the nontrivial automorphism it carries. Can we create a moduli space which parameterizes rational maps with a choice of automorphism, and would the analogue of $\mathcal{A}_d$ in this moduli space be a Zariski closed set?

Thanks for Listening!

Questions?

References

- P. Doyle and C. McMullen, *Solving the quintic by iteration*, Acta Arith. **163** (1989), no. 3–4, 151–180.
- Xander Faber, *Finite p-irregular subgroups of PGL2(k)*, arxiv.org/1112.1999 (2012).
- Masayo Fujimura and Kiyoko Nishizawa, *The real multiplier coordinate space of the quartic polynomials*, pp. 61–69, Yokohama Publ., 2007.
- Benjamim Hutz and Michael Tepper, *Multiplier spectra and the moduli space of degree 3 morphisms on P1*, JP Journal of Algebra, Number Theory and Applications **29** (2013), no. 2, 189–206.
- Curtis McMullen, *Families of rational maps and iterative root-finding algorithms*, Ann. of Math. **125** (1987), no. 3, 467–493.
- J. Milnor, *Geometry and dynamics of quadratic rational maps*, Experiment. Math. **2** (1993), no. 1, 37–83.
- Nikita Miasnikov, Brian Stout, and Phillip Williams, *Automorphism loci for the moduli space of rational maps*, arxiv:1408.5655 (2014).
- Joseph H. Silverman, *The space of rational maps on P1*, Duke Math. J. **94** (1998), 41–118.
- Llyod W. West, *The moduli space of cubic rational maps*, arXiv:1408.3247 (2014).