Arboreal Galois representations of PCF quadratic polynomials

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Arboreal Galois Representations

Let

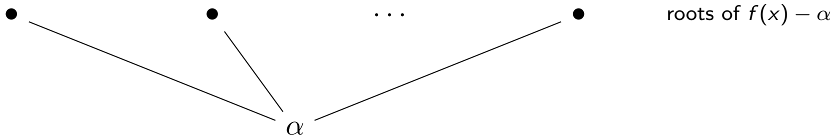
- K be a field,
- $f(x) \in K(x)$ with degree $d > 1$ and $\alpha \in K$,
- $f^n(x) = \underbrace{f \circ f \circ \cdots \circ f}_{n}(x)$, the n-th iterate of f,
- $f^{-n}(\alpha) = \{\beta \in \bar{K} \;:\; f^n(\beta) = \alpha\}$,
- $K_n = K\big(f^{-n}(\alpha)\big)$, and $K_\infty = \varprojlim K_n$.
- Suppose $f^n(x) - \alpha$ is separable $\forall n \in \mathbb{N}$, and consider

$$G_n = \mathrm{Gal}(K_n/K), \qquad G_\infty = \mathrm{Gal}(K_\infty/K).$$

We can see G_∞ can be embedded in $\mathrm{Aut}(T_d)$, the automorphism group of a rooted tree.

Tree Diagram for Roots of $f^n(x) - \alpha$, $d = \deg f(x)$

- Draw roots of $f^n(x) - \alpha$ at n-th level.
- Connect β and γ if $f(\beta) = \gamma$.



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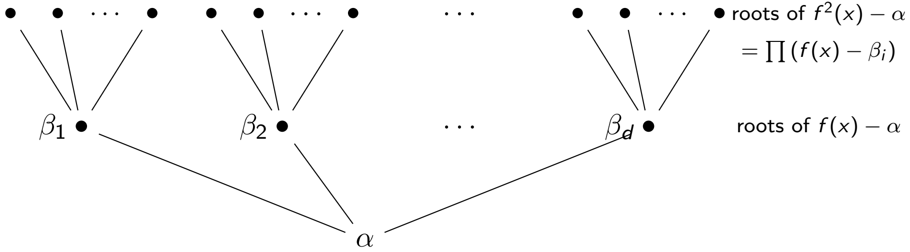

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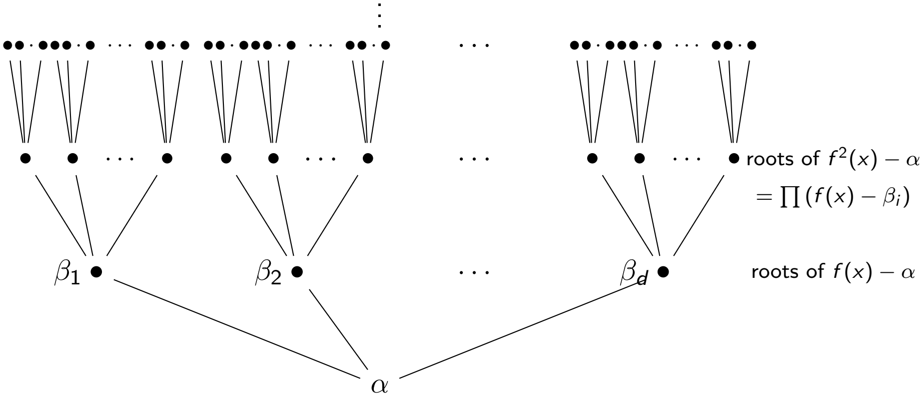
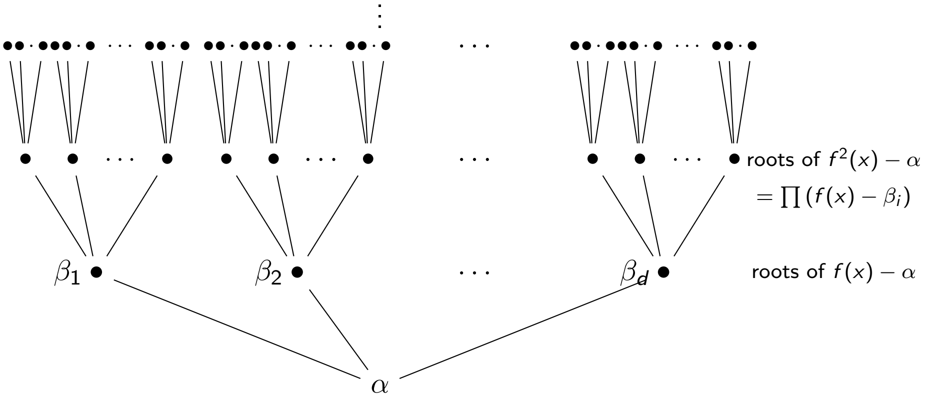


Tree Diagram for Roots of $f^n(x) - \alpha$



- $G_n = \mathrm{Gal}\left(K_n/\mathbb{Q}\right)$ acts on the tree up to the n-th level by permuting the branches.
- G_n is isomorphic to a subgroup of $Aut(T_{d,n}) \cong [S_d]^n$.

Arithmetic and Geometric Iterated Monodromy Groups

Let

- t be transcendental over K,
- $f(x) \in K(x)$,
- $K_{\infty,t} = \varprojlim K(f^{-n}(t))$, and
- $G^{arith} = \mathrm{Gal}(K_{\infty,t}/K(t))$.

Note, for any $\alpha \in K$, G_∞ is a subgroup of G^{arith}.

- Let $L = \bar{K} \cap K_\infty$ and
- $G^{geom} = \mathrm{Gal}(K_\infty/L(t))$

Then G^{geom} is a normal subgroup of G^{arith}, and

$$G^{arith}/G^{geom} \cong \mathrm{Gal}(L/K).$$

Our Focus: PCF Quadratic Polynomials

We say f is **post-critically finite** (PCF) if $\bigcup_{\substack{n\geq 1 \\ f'(c)=0}} f^n(c)$ is finite.

If f is PCF, then $[\mathrm{Aut}(T_d) : G^{arith}] = \infty$.

Two natural questions arise:

- For any PCF polynomial f defined over a number field K, can we describe G^{arith}, G^{geom}, $L = K_{\infty,t} \cap \bar{K}$?
- Further, for α in K, can we give sufficient conditions ensuring $G_\infty \cong G^{arith}$?

We consider these questions for quadratic polynomials.

Previous Results

- In 2013, Pink gave description of G^{arith}, G^{geom}, $L = K_{\infty,t} \cap \bar{K}$ for any PCF quadratic polynomial using theory of self similar groups and algebraic geometry.
- In 2019, Ahmad, Benedetto, Cain, Carroll, Fang gave another (very explicit) description of G^{arith} (the arithmetic basilica group), G^{geom}, $L = K_{\infty,t} \cap \bar{K}$ for $f(x) = x^2 - 1$ and sufficient conditions on α ensuring $G_\infty \cong G^{arith}$.

We want to give unified explicit descriptions for all PCF quadratic polynomials.

Some details...

Let $f(x) = x^2 + c \in K$.

- Consider the geometric case

$$G_n^{geom} = \mathrm{Gal}(\bar{K}(f^{-n}(t))/\bar{K}(t)).$$

- Primes of $\bar{K}[t]$ that ramify in $\bar{K}(f^n(t))$ have the form

$$f^m(0) - t$$

  for $m \leq n$.

- Suppose the forward orbit of 0 has total length r and tail length s.

$$0 \longrightarrow f(0) \longrightarrow \cdots \longrightarrow f^s(0) \to f^{s+1}(0) \longrightarrow \cdots \longrightarrow f^r(0)$$

Inertia Groups

What do the inertia groups look like?

Let $\mathfrak{q}_i$ be any prime lying above $(f^i(0) - t)$. Let $<\sigma_i>= I(\mathfrak{q}_i|(f^i(0) - t))$.

Key Fact: The cycle type of the action of σ_i on the roots of $f^n(x) - t$ corresponds to the factorization of

$$f^n(x) - f^i(0) \equiv f^n(x) - t \mod (f^i(0) - t).$$

Note: $f(x) - f(0) = x^2$ and

$$\begin{aligned} f^n(x) - f^i(0) &= (f^{n-1}(x)^2 + c) - (f^{i-1}(0)^2 + c) \\ &= (f^{n-1}(x) - f^{i-1}(0))(f^{n-1}(x) + f^{i-1}(0)) \end{aligned}$$

Example: $s = 2$, $r = 4$

$$0 \longrightarrow f(0) \longrightarrow f^2(0) \longrightarrow f^3(0) \longrightarrow f^4(0)$$

$$f^4(0) \curvearrowright f^3(0)$$

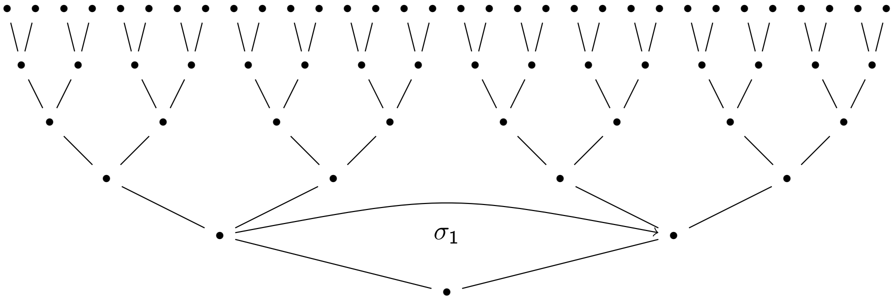
One can quickly check:

- $f^4(0) = -f^2(0)$
- $f^j(x)$, $f^j(x) + f(0)$, $f^j(x) + f^3(0)$ are separable for every j
- $f^j(x) + f^2(0)$ is separable for $j \leq 3$

Example: $s = 2$, $r = 4$

$$< \sigma_1 >= I(q_1 | (f(0) - t))$$

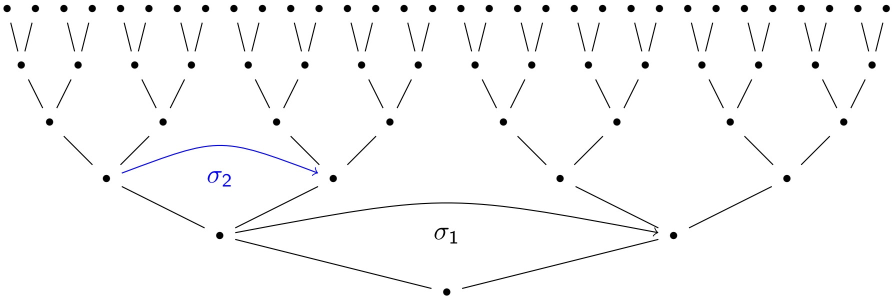
$$f(x) - f(0) = x^2$$
$$f^2(x) - f(0) = (f(x))^2$$
$$f^n(x) - f(0) = (f^{n-1}(x))^2$$



Example: $s = 2$, $r = 4$

$$< \sigma_2 >= I(q_2|(f^2(0) - t))$$

$$\begin{aligned} f(x) - f^2(0) &= (x - f(0))(x + f(0)) \\ f^2(x) - f^2(0) &= x^2(f(x) + f(0)) \\ f^n(x) - f^2(0) &= (f^{n-2}(x))^2(f^{n-1}(x) + f(0)) \end{aligned}$$

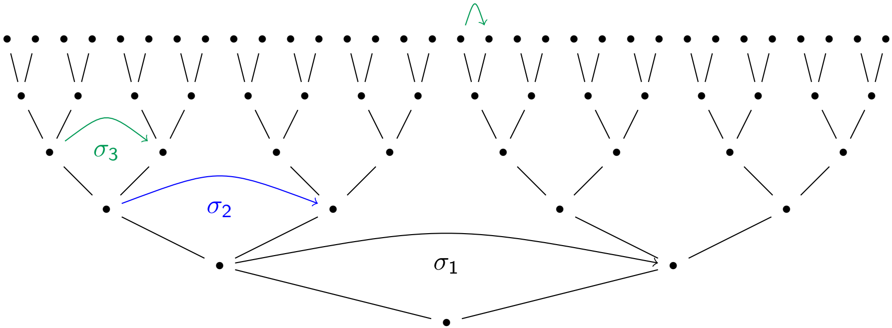


Example: $s = 2$, $r = 4$

$< \sigma_3 >= I(q_3|(f^3(0) - t))$

$$\begin{aligned} f^3(x) - f^3(0) &= x^2(f(x) + f(0))(f^2(x) + f^2(0)) \\ f^4(x) - f^3(0) &= (f(x))^2(f^2(x) + f(0))(f^3(x) + f^2(0)) \\ f^5(x) - f^3(0) &= (f^2(x))^2(f^3(x) + f(0))(f^4(x) + f^2(0)) \\ &= (f^2(x))^2(f^3(x) + f(0))(f^4(x) - f^4(0)) \end{aligned}$$

σ_3 acts on the r.h.s like σ_4 acts on the whole tree.



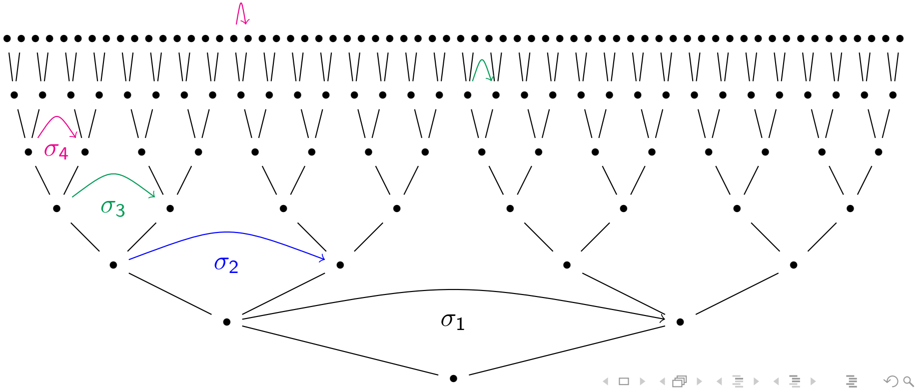
Example: $s = 2,\ r = 4$

$$<\sigma_4> = I(q_4|(f^4(0) - t))$$

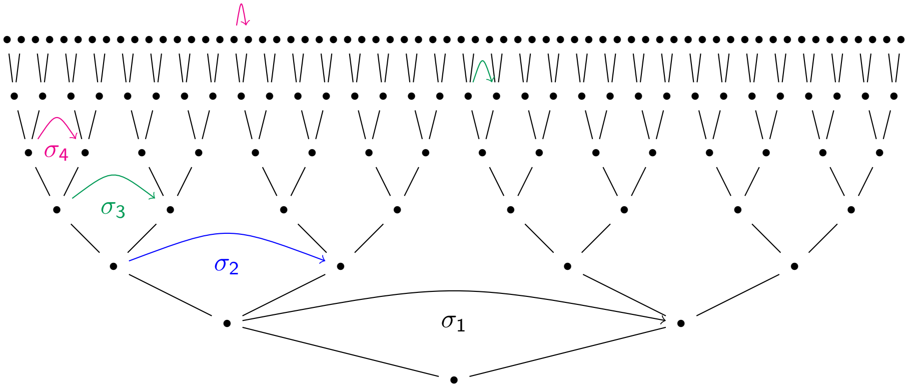
$$f^4(x) - f^4(0) = x^2(f(x) + f(0))(f^2(x) + f^2(0))(f^3(x) + f^3(0))$$

$$f^5(x) - f^4(0) = (f(x))^2(f^2(x) + f(0))(f^3(x) + f^2(0))(f^4(x) + f^3(0))$$

$$\begin{aligned} f^6(x) - f^4(0) &= (f^2(x))^2(f^3(x) + f(0))(f^4(x) + f^2(0))(f^5(x) + f^3(0)) \\ &= (f^2(x))^2(f^3(x) + f(0))(f^4(x) - f^4(0))(f^5(x) + f^3(0)) \end{aligned}$$



Example: $s = 2$, $r = 4$



Let $\tau \in G^{geom}$ and y be any node in the tree. The actions of τ on $f^{-2}(y)$ and $f^{-4}(-y)$ have the same parity. This condition completely describes G^{geom}.

Example: $s = 2$, $r = 4$

What about G^{arith}?

Using self-similarity and a size argument,

$$|G_4^{arith}/G_4^{geom}| = 1$$

$$|G^{arith}/G^{geom}| \leq 2 \text{ for } n \geq 5$$

Further, for any node y in the tree let $f^{-2}(y) = \{\pm\alpha_1, \pm\alpha_2\}$ and $f^{-4}(-y) = \{\pm\beta_1, \pm\beta_2, \cdots \pm \beta_8\}$. Then

$$\left(\frac{\alpha_1\alpha_2}{\beta_1\beta_2\ldots\beta_8}\right)^2 = \frac{f^2(0) - y}{f^4(0) + y} = -1,$$

so

$$\frac{\alpha_1\alpha_2}{\beta_1\beta_2\ldots\beta_8} = i.$$

Hence, $i \in K(f^{-5}(t))$ and if $\tau(i) = -i$ the actions of τ on $f^{-2}(y)$ and $f^{-4}(-y)$ have opposite parity.

Example: $s = 2$, $r = 4$

What about algebraic base points?

Let $\alpha \in K$, then G_∞ is isomorphic to a subgroup of G^{arith}.

Idea:

If $[K(i, \sqrt{f(0) - \alpha}, \dots, \sqrt{f^4(0) - \alpha}) : K] = 2^5$, we have primes ramifying whose inertia groups are similar to the geometric case so

$$G_\infty \cong G^{arith}.$$

Thank you