# Algebraic preperiodic points of entire transcendental functions

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# History

Let $\mathbb{A}$ denote the field of algebraic numbers, i.e., the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, and let $\mathbb{T} = \mathbb{C} \setminus \mathbb{A}$ be the set of transcendental numbers.

- Liouville, 1844: $\mathbb{T}$ is nonempty. For example, $\sum_{n=1}^{\infty} 10^{-n!} \in \mathbb{T}$.
- Hermite, 1873: The number $e$ is transcendental.
- Lindemann, 1882: if $x \in \mathbb{A}$ is nonzero, then $e^x \in \mathbb{T}$.

Some consequences of Lindemann's theorem:

- $\pi \in \mathbb{T}$. (If $\pi \in \mathbb{A}$, then $i\pi \in \mathbb{A}$, so $e^{i\pi} \in \mathbb{T}$.)
- If $x \in \mathbb{A}$ is nonzero, then $\sin(x), \cos(x) \in \mathbb{T}$.

# Stäckel functions

**Question (Weierstrass)**

Is it the case that every entire transcendental function will map ‘most’ algebraic numbers to transcendental numbers?

(Recall that if $D \subseteq \mathbb{C}$, a map $f : D \to \mathbb{C}$ is called **algebraic** if there exists a nonzero polynomial $P \in \mathbb{C}[x, y]$ such that $P(z, f(z)) = 0$ for all $z \in D$. If $f$ is not algebraic, it is called **transcendental**.)

- Weierstrass, 1886: There exists an entire transcendental function $f$ such that $f(\mathbb{Q}) \subseteq \mathbb{Q}$.
- Stäckel, 1895: There exists an entire transcendental function $f$ such that $f(\mathbb{A}) \subseteq \mathbb{A}$.

By a **Stäckel function** we mean an entire transcendental function $f$ satisfying $f(\mathbb{A}) \subseteq \mathbb{A}$.

# Construction of Stäckel functions

Let $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \ldots$ be an enumeration of all the Galois orbits of nonzero algebraic numbers. Define polynomials $P_n(z)$ for $n \geq 1$ by

$$P_n(z) = \prod_{\alpha \in \mathcal{O}_1 \cup \mathcal{O}_2 \cup \cdots \cup \mathcal{O}_n} (z - \alpha).$$

Note that:

- $P_n \in \mathbb{Q}[z]$ and $P_n$ divides $P_{n+1}$ for all $n$.
- For every $\alpha \in \mathbb{A}$ there exists $M$ such that $P_M(\alpha) = 0$, and thus $P_n(\alpha) = 0$ for all $n \geq M$.

Choose rational numbers $0 < \epsilon_n < \frac{1}{L(P_n)n!}$, where $L$ represents length.

Define $f(z) = \sum_{n=1}^{\infty} \epsilon_n \cdot z^{\delta_n} \cdot P_n(z)$ for appropriate $\delta_n$. Then $f$ is entire. Also, $f$ is not a polynomial, so it is transcendental.

Finally, $f(\mathbb{A}) \subseteq \mathbb{A}$: given $\alpha \in \mathbb{A}$, we have $P_n(\alpha) = 0$ for $n \geq M$, so $f(\alpha) = \sum_{n=1}^{M-1} \epsilon_n \cdot \alpha^{\delta_n} \cdot P_n(\alpha) \in \mathbb{A}$.

# Dynamics of Stäckel functions

If $f$ is a Stäckel function, we can regard $f$ as a dynamical system on $\mathbb{A}$.

**Question**

Let $\Pi(f)$ denote the directed graph of preperiodic points of $f$ in $\mathbb{A}$; we will call $\Pi(f)$ the **portrait** of $f$. What can the structure of $\Pi(f)$ be?

Restrictions in the case where $f(z) \in \mathbb{A}[z]$:

- For every positive integer $n$, the set of $n$-cycles in $\Pi(f)$ is finite.
- The set of indegrees of the vertices of $\Pi(f)$ is bounded.

In the transcendental case we are not aware of any such restrictions.

**Conjecture**

Let $\Gamma$ be a countable directed graph such that every vertex in $\Gamma$ has outdegree 1 and every vertex eventually reaches a cycle. Then there exists a Stäckel function $f$ such that $\Pi(f) \cong \Gamma$.

**Example:** Suppose $\Gamma$ is a graph consisting of infinitely many $n$-cycles for every $n \geq 1$. We can construct a Stäckel function $f$ such that $\Pi(f) \cong \Gamma$.

Let $g$ be any Stäckel function constructed as shown earlier.

**Theorem (Bergweiler, 1991)**

If $g$ is an entire transcendental function, then $g$ has infinitely many complex points of period $n$ for every $n \geq 2$.

**Step 1:** Using the Great Picard Theorem we can also obtain infinitely many fixed points by modifying $g$ slightly: for some $c \in \mathbb{A}$, the equation $g(z) - z = c$ will have infinitely many solutions. Hence the function $f_0(z) := g(z) - c$ has infinitely many fixed points. Note that $f_0$ is a Stäckel function, so Bergweiler's theorem applies to $f_0$, and therefore $f_0$ has infinitely many complex points of every period.

# Dynamics of Stäckel functions

**Step 2:** Modify $f_0$ so that its $n$-periodic points will be algebraic. The main tool used for this is Rouché's theorem, the key idea being that if $f_0$ has some number of $n$-cycles inside a disk $D$, then a slight perturbation of $f_0$ will have the same number of $n$-cycles in $D \cap \mathbb{A}$.)

Using this idea we construct a sequence of functions $f_0, f_1, f_2, \ldots$, and we show that the limit function $f(z) = \lim_{n\to\infty} f_n(z)$ is a Stäckel function having infinitely many $n$-cycles in $\mathbb{A}$ for every $n \geq 1$. Thus $\Pi(f) \cong \Gamma$.

**Conjecture**

Let $\Gamma$ be a countable directed graph such that every vertex in $\Gamma$ has outdegree 1 and every vertex eventually reaches a cycle. Then there exists a Stäckel function $f$ such that $\Pi(f) \cong \Gamma$.

# Quasi-polynomial functions

Stäckel (1902) also constructed a transcendental function $f$, analytic on a disk $D$ centered at 0, such that $f(\mathbb{A} \cap D) \subseteq \mathbb{A}$ and $f(\mathbb{T} \cap D) \subseteq \mathbb{T}$.

**Question (Mahler, 1976)**

Can we replace $D$ with $\mathbb{C}$?

**Theorem (Marques–Moreira, 2016)**

There exist uncountably many entire transcendental functions $f$ such that $f(\mathbb{A}) = \mathbb{A}$ and $f(\mathbb{T}) = \mathbb{T}$.

Note the similarity with polynomial functions: if $f(z) \in \mathbb{A}[z] \setminus \mathbb{A}$, then $f$ has all of the above properties except that it is algebraic.

An additional similarity: for both types of functions, if the coefficients of the Taylor series of $f(z)$ lie in a field $K \subseteq \mathbb{A}$, then $\sigma(f(x)) = f(\sigma(x))$ for every $\sigma \in \mathrm{Gal}(\mathbb{A}/K)$ and every $x \in \mathbb{A}$.

For these reasons, functions $f$ as in the theorem will be called **quasi-polynomial**.

# Dynamics of quasi-polynomial functions

If $f$ is a quasi-polynomial function, we can regard $f$ as a dynamical system on either $\mathbb{A}$ or $\mathbb{T}$.

**Question**

With $\Pi(f)$ defined as before, what can the structure of $\Pi(f)$ be?

(It would also be interesting to study the dynamics on $\mathbb{T}$.)

**Conjecture**

Let $\Gamma$ be a countable directed graph such that every vertex in $\Gamma$ has outdegree 1 and every vertex eventually reaches a cycle. Then there exists a quasi-polynomial map $f$ such that $\Pi(f) \cong \Gamma$.

# Main result

**Theorem (K.–Marques–Moreira)**

Let $\Gamma$ be a countable graph consisting of only cycles. Then there exists a quasi-polynomial function $f$ such that $\Pi(f) \cong \Gamma$.

In fact, we prove a stronger result: starting with any entire transcendental function $g$, we can construct a "nearby" quasi-polynomial function $f$ with the required property. More precisely, given $\epsilon > 0$, if

$$g(z) = \sum_{n=0}^{\infty} a_n z^n,$$

we construct

$$f(z) = \sum_{n=0}^{\infty} b_n z^n$$

such that $f$ is quasi-polynomial, $\Pi(f) \cong \Gamma$, and $|a_n - b_n| < \epsilon$ for all $n$.