Moduli spaces for dynamical systems with level structure

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Example to get started

Classify degree-2 morphisms on $\mathbb{P}^1$ realizing the following portrait:

$$\mathcal{P}: \quad \alpha \longrightarrow \beta \rightleftarrows \gamma$$

This portrait is realized by

$$\begin{aligned}(f;\ \alpha,\beta,\gamma) = {} & (x^2-1;\ 1,0,-1),\\ & (x^2-2x+1;\ 2,1,0),\\ & (x^2-3;\ -1,-2,1),\\ & \left(\frac{z-1}{2z^2-1};\ \infty,0,1\right),\ldots\end{aligned}$$

Parameter space of endomorphisms

$\mathrm{End}_d^N := \{\text{degree-}d \text{ morphisms } f : \mathbb{P}^N \to \mathbb{P}^N\}$

$f = (f_0(X_0, \ldots, X_N) : \cdots : f_N(X_0, \ldots, X_N))$

f_i : homogeneous, degree d, no common (nontrivial) zeroes

Setting $M = M(N, d) = \#(\text{coefficients of } f) - 1$, we embed

$$\mathrm{End}_d^N \hookrightarrow \mathbb{P}^M.$$

More precisely,

$$\mathrm{End}_d^N = \mathbb{P}^M \setminus (\text{Resultant} = 0)$$

is an affine variety.

Moduli space of endomorphisms

Morphisms $f, g : \mathbb{P}^N \to \mathbb{P}^N$ are **equivalent** if there exists $\gamma \in \mathrm{PGL}_{N+1} = \mathrm{Aut}(\mathbb{P}^N)$ such that

$$g = f^\gamma := \gamma^{-1} \circ f \circ \gamma.$$

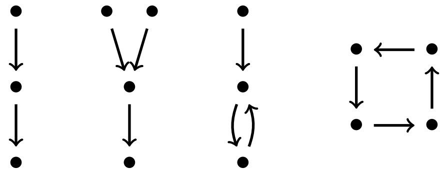
The **moduli space** of degree-d endomorphisms on $\mathbb{P}^N$, denoted $\mathcal{M}_d^N$, is the space of *equivalence classes* of endomorphisms.

Theorem (Silverman, 1998; Petsche-Szpiro-Tepper, 2007; Levy, 2010)

The moduli space $\mathcal{M}_d^N = \mathrm{End}_d^N /\!\!/ \mathrm{PGL}_{N+1}$ *exists as a geometric quotient (in the sense of GIT).*

Portraits

Informally, a portrait is a directed graph whose vertices have outdegree 0 or 1 and whose edges have positive integer weights.



Formally, a portrait is a tuple $(\mathcal{V}, \mathcal{W}, \Phi, \epsilon)$ such that

- $\mathcal{V}$ is a finite set; (vertices)
- $\mathcal{W} \subseteq \mathcal{V}$; (vertices with out-edges)
- $\Phi : \mathcal{W} \to \mathcal{V}$; (directed edges)
- $\epsilon : \mathcal{W} \to \mathbb{N}$. (multiplicities)

Parameter space of morphisms with level structure

Let $\mathcal{P} = (\mathcal{V}, \mathcal{W}, \Phi, \epsilon)$ be a portrait. Write $\mathcal{V} = \{1, \ldots, n\}$.

$\mathrm{End}_d^N[\mathcal{P}]$: the space of tuples

$$(f;\ P_1, \ldots, P_n) \in \mathrm{End}_d^N \times \left(\mathbb{P}^N\right)^n$$

such that

- $f(P_i) = P_{\Phi(i)}$ for all $i \in \mathcal{W}$;
- $e_f(P_i) \geq \epsilon(i)$ for all $i \in \mathcal{W}$; and
- $P_i \neq P_j$ for all $1 \leq i < j \leq n$.

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such that

- $f(P_i) = P_{\Phi(i)}$ for all $i \in \mathcal{W}$; (closed condition)
- $e_f(P_i) \geq \epsilon(i)$ for all $i \in \mathcal{W}$; and (closed condition)
- $P_i \neq P_j$ for all $1 \leq i < j \leq n$. (open condition)

Thus $\mathrm{End}_d^N[\mathcal{P}]$ is naturally a subvariety of

$$\mathrm{End}_d^N \times \left(\mathbb{P}^N\right)^n \subset \mathbb{P}^M \times \left(\mathbb{P}^N\right)^n.$$

Moduli space of morphisms with level structure

PGL_{N+1} acts on $\mathrm{End}_d^N[\mathcal{P}]$:

$$(f;\ P_1,\ldots,P_n)^\gamma := \left(f^\gamma;\ \gamma^{-1}(P_1),\ldots,\gamma^{-1}(P_n)\right).$$

Theorem (D-Silverman, 2019)

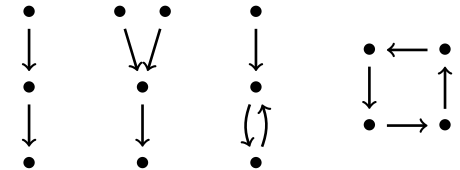
The moduli space $\mathcal{M}_d^N[\mathcal{P}] = \mathrm{End}_d^N[\mathcal{P}] /\!\!/ \mathrm{PGL}_{N+1}$ *exists as a geometric quotient (in the sense of GIT).*

Moduli space of morphisms with level structure

Proof idea.

(a) The *closure* of $\mathrm{End}_d^N[\mathcal{P}] \subset \mathrm{End}_d^N \times \left(\mathbb{P}^N\right)^n$ admits a finite morphism onto $\mathrm{End}_d^N \times \left(\mathbb{P}^N\right)^m$ for some m.

$n = 14$



(b) Using Mumford's numerical criterion, show that $\mathrm{End}_d^N \times \left(\mathbb{P}^N\right)^m$ is in the GIT stable locus of $\mathbb{P}^M \times \left(\mathbb{P}^N\right)^m$ *for an appropriate line bundle*.

$\square$

Moduli space of morphisms with level structure

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$m = 2$ • •

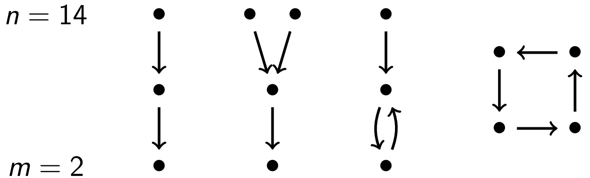
(b) Using Mumford's numerical criterion, show that $\mathrm{End}_d^N \times \left(\mathbb{P}^N\right)^m$ is in the GIT stable locus of $\mathbb{P}^M \times \left(\mathbb{P}^N\right)^m$ *for an appropriate line bundle*.

□

Moduli space of morphisms with level structure

Proof idea.

(a) The *closure* of $\mathrm{End}_d^N[\mathcal{P}] \subset \mathrm{End}_d^N \times (\mathbb{P}^N)^n$ admits a finite morphism onto $\mathrm{End}_d^N \times (\mathbb{P}^N)^m$ for some m.



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□

Dimension

Theorem (DS)

Suppose that $\mathcal{P}$ is unweighted or that $\mathcal{P}$ is weighted and $N = 1$.
If $\mathrm{End}_d^N[\mathcal{P}] \neq \emptyset$, then $\mathrm{End}_d^N[\mathcal{P}]$ and $\mathcal{M}_d^N[\mathcal{P}]$ have the expected dimension.

The proof uses a result of Fakhruddin on generic morphisms on $\mathbb{P}^N$; for weighted portraits and $N = 1$ we use Thurston transversality.

Note: For unweighted portraits, there is a simple condition on $\mathcal{P}$ to determine whether $\mathrm{End}_d^N[\mathcal{P}] \neq \emptyset$.

Uniform boundedness

A point P is **preperiodic** for f if the sequence

$$P, f(P), f(f(P)), \ldots$$

is eventually periodic.

Uniform Boundedness Conjecture (Morton-Silverman, 1994)

Let $d \geq 2$ and $n, N \geq 1$.

There is a bound $B(N, d, n)$ such that if $f : \mathbb{P}^N \to \mathbb{P}^N$ is a degree-d morphism defined over a degree-n number field K, then

$$\#\{K\text{-rational preperiodic points for } f\} \leq B(N, d, n).$$

Uniform boundedness

A portrait $\mathcal{P}$ is **preperiodic** if every vertex has an out-edge (i.e., if $\mathcal{W} = \mathcal{V}$).

Moduli Boundedness Conjecture

Let $d \geq 2$ and $n, N \geq 1$.

There is a bound $C(N, d, n)$ such that if $\mathcal{P}$ is a preperiodic portrait with $|\mathcal{V}| \geq C(N, d, n)$ and K is a number field of degree n, then

$$\mathcal{M}_d^N[\mathcal{P}](K) = \emptyset.$$

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Theorem (DS)

UBC $\iff$ MBC.

Uniform boundedness: $z^2 + c$

Let $f(z) = z^2 + c$ with $c \in \mathbb{Q}$.

Theorem (Morton, 1998)

f has no rational points of period 4.

Theorem (Flynn-Poonen-Schaefer, 1997)

f has no rational points of period 5.

Theorem (Stoll, 2008)

f has no rational points of period 6, assuming standard conjectures for the Jacobian of a certain genus 4 curve.

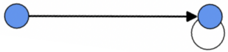
Theorem (Poonen, 1998)

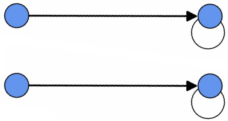
If f has no rational points of period greater than 3, then f has at most 9 rational preperiodic points.

Uniform boundedness: $z^4 + c$

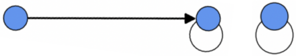
(Joint work with Grip, Rachfal, Schwager, Torrence; Summer@ICERM 2019)

Question: Which portraits can be realized by $z^4 + c$ over $\mathbb{Q}$?

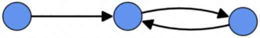




Infinitely many $c \in \mathbb{Q}$



$c = 0$



$c = -1$

Finitely many $c \in \mathbb{Q}$

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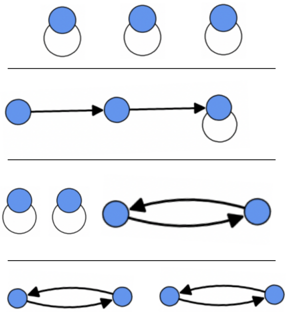
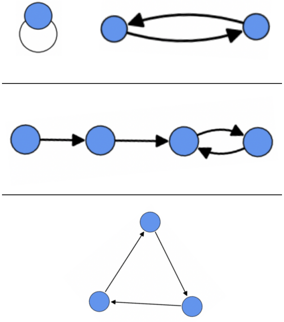


No $c \in \mathbb{Q}$



No $c \in \mathbb{Q}$ *up to large height*

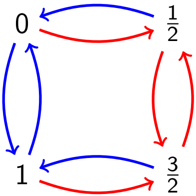
Multi-portraits for dynamical semigroups

(Joint work with Blum, Hyde, Kelln, Talbott, Weinreich; Summer@ICERM 2019)

Example:

$f(z) = 2z^2 - 3z + 1$

$g(z) = -2z^2 + 3z + \frac{1}{2}$



If $\mathcal{P}$ is a *multi-portrait* with an m-edge-coloring, then one can construct the appropriate moduli space

$$\mathcal{M}_d^N[\mathcal{P}] = \mathrm{End}_d^N[\mathcal{P}] \mathbin{/\!\!/} \mathrm{PGL}_{N+1}$$

as a geometric quotient.

Thank you!