A Dynamical Pairing Between Two Rational Maps

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joint work with:
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arxiv: 0911.1875

AMS Special Session on Arithmetic and Nonarchimedean Dynamics
San Francisco 2010
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- Let $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map, of degree at least two, defined over $K$. 

Let $h_{\varphi} : \mathbb{P}^1(\bar{K}) \to \mathbb{R}$ be the Call-Silverman canonical height function:

$$h_{\varphi}(x) := \lim_{\ell \to +\infty} h(\varphi^\ell(x)) / \deg(\varphi^\ell)$$

**Properties:**

- $h_{\varphi}(x) = h(x) + O(1)$
- $h_{\varphi}(\varphi(x)) = \deg(\varphi) h_{\varphi}(x)$
- $h_{\varphi}(x) \geq 0$
- $h_{\varphi}(x) = 0$ if and only if $x$ is $\varphi$-preperiodic.
Let $K$ be a number field. Let $h : \mathbb{P}^1(\overline{K}) \to \mathbb{R}$ be the absolute Weil (or “naive”) height function. Let $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map, of degree at least two, defined over $K$. Let $h_\varphi : \mathbb{P}^1(\overline{K}) \to \mathbb{R}$ be the Call-Silverman canonical height function:

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    h_\varphi(x) &= h(x) + O(1) \\
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    h_\varphi(x) &\geq 0
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3. $h_\varphi(x) \geq 0$
4. $h_\varphi(x) = 0$ if and only if $x$ is $\varphi$-preperiodic.
Let $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ and $\psi : \mathbb{P}^1 \to \mathbb{P}^1$ be two rational maps, each having degree at least two, defined over $K$. 

The Arakelov-Zhang pairing associated to $\varphi$ and $\psi$ is a real number $\langle \varphi, \psi \rangle$. It is symmetric: $\langle \varphi, \psi \rangle = \langle \psi, \varphi \rangle$; and nonnegative: $\langle \varphi, \psi \rangle \geq 0$. Intuitively, $\langle \varphi, \psi \rangle$ can be viewed as a measure of the "dynamical distance" between $\varphi$ and $\psi$. Our definition of $\langle \varphi, \psi \rangle$ uses local analytic machinery at each place. An equivalent (and more standard) definition of $\langle \varphi, \psi \rangle$ can be given using arithmetic intersection theory.
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Main Results

The value of $\langle \varphi, \psi \rangle$ is closely related to the canonical heights of points which are small with respect to at least one of the rational maps $\varphi$ and $\psi$. 

Theorem 1

Let $\{x_n\}$ be a sequence of distinct points in $P_1(\bar{K})$ such that $h_\psi(x_n) \to 0$. Then $h_\varphi(x_n) \to \langle \varphi, \psi \rangle$. 

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**Theorem 2**

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\langle \varphi, \psi \rangle = \lim_{n \to +\infty} \frac{1}{\deg(\psi)^n + 1} \sum_{\psi^n(x) = x} h_\varphi(x),
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(d) \( \text{PrePer}(\varphi) \cap \text{PrePer}(\psi) \) is infinite;
(e) \( \lim \inf_{x \in \mathbb{P}^1(\bar{K})} (h_\varphi(x) + h_\psi(x)) = 0 \).
For each place $v$ of $K$, we have the following local objects:

- $C_v$, the $v$-adic complex field;
- $P_1^v$, the $v$-adic Berkovich projective line;
- $\Delta$, the measure-valued Laplacian on $P_1^v$.

The pairing $\langle \phi, \psi \rangle$ is defined as a sum of local terms of the form $-\int \lambda \phi, v \Delta \lambda \psi, v$.

Self-adjoint property of $\Delta$ $\Rightarrow$ Symmetric property $\langle \phi, \psi \rangle = \langle \psi, \phi \rangle$. 
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The pairing $\langle \varphi, \psi \rangle$ is defined as a sum of local terms of the form $-\int \lambda \varphi, v \Delta \lambda \psi, v$. The self-adjoint property of $\Delta$ implies the symmetric property $\langle \varphi, \psi \rangle = \langle \psi, \varphi \rangle$. 

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For each place $\nu$ of $K$, we have the following local objects:

- $\mathbb{C}_\nu$, the $\nu$-adic complex field;
- $P^1_\nu$, the $\nu$-adic Berkovich projective line;
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For each place $v$ of $K$, we have the following local objects:

- $\mathbb{C}_v$, the $v$-adic complex field;
- $P^1_v$, the $v$-adic Berkovich projective line;
- $\Delta$, the measure-valued Laplacian on $P^1_v$;
- $\lambda_{\varphi,v} : P^1_v \to \mathbb{R} \cup \{+\infty\}$, a canonical local height function associated with the rational map $\varphi$.

The pairing $\langle \varphi, \psi \rangle$ is defined as a sum of local terms of the form $-\int \lambda_{\varphi,v} \Delta \lambda_{\psi,v}$.

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- Self adjoint property of $\Delta \Rightarrow$ Symmetric property $\langle \varphi, \psi \rangle = \langle \psi, \varphi \rangle$
All of our main results follow from applying local dynamical equidistribution results (Baker and Rumely; Chambert-Loir; Favre and Rivera-Letelier; Ljubich) for small points, testing against functions of the form

\[ x \mapsto \lambda_{\varphi,v}(x) - \lambda_{\psi,v}(x). \]  

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The idea behind the proofs

All of our main results follow from applying local dynamical equidistribution results (Baker and Rumely; Chambert-Loir; Favre and Rivera-Letelier; Ljubich) for small points, testing against functions of the form

\[ x \mapsto \lambda_{\varphi,v}(x) - \lambda_{\psi,v}(x). \] (1)

Key technical point: we may take the local height functions \( \lambda_{\varphi,v} \) and \( \lambda_{\psi,v} \) to have their singularities at the same point, which implies that (1) is continuous.
Examples

- Let \( \sigma : \mathbb{P}^1 \to \mathbb{P}^1 \) denote the squaring map \( \sigma(x) = x^2 \).
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- In this case the canonical height $h_\sigma$ is the same as the naive height $h$.
- So given an arbitrary rational map $\psi : \mathbb{P}^1 \to \mathbb{P}^1$, we may view $\langle \sigma, \psi \rangle$ as a measure of how far the canonical height $h_\psi$ is from the naive height $h$. 
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So given an arbitrary rational map $\psi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$, we may view $\langle \sigma, \psi \rangle$ as a measure of how far the canonical height $h_\psi$ is from the naive height $h$.

We give estimates on $\langle \sigma, \psi \rangle$ as $\psi$ varies in certain families.
Example 4

Let $\psi(x) = \alpha - (\alpha - x)^2$ for $\alpha \in K$. Here $\psi$ is the squaring map conjugated by the involution $x \mapsto \alpha - x$. Then

$$\langle \sigma, \psi \rangle = h(\alpha) + O(1).$$
Example 4

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When $\alpha = 1$, $\langle \sigma, \psi \rangle$ is equal to the Mahler measure of the polynomial $1 + x + y$, and therefore also (by a calculation of Smyth) equal to the value at $s = 2$ of a certain Dirichlet $L$-function.
Example 5

Let $\psi(x) = x^2 + c$ for $c \in K$. Then

$$\langle \sigma, \psi \rangle = \frac{1}{2} h(c) + O(1).$$
Example 6

Let \( \psi(x) = \frac{(x^2 + ab)^2}{4x(x - a)(x + b)} \) be the Lattès map associated to the doubling map on the elliptic curve \( E \) given by the Weierstrass equation \( y^2 = x(x - a)(x + b) \), where \( a \) and \( b \) are positive integers. Then

\[
\langle \sigma, \psi_E \rangle = \log \sqrt{ab} + O(1).
\]
Using our main results and the dynamical Mahler formula of Szpiro-Tucker, we prove the following upper bound on the difference between the canonical and naive heights.

**Theorem 7**

Let $\sigma: \mathbb{P}^1 \to \mathbb{P}^1$ be the squaring map $\sigma(x) = x^2$, and let $\psi: \mathbb{P}^1 \to \mathbb{P}^1$ be an arbitrary map of degree at least two defined over $K$. Then

$$h(\psi)(x) - h(x) \leq \langle \sigma, \psi \rangle + h(\psi)(\infty) + \log 2$$

for all $x \in \mathbb{P}^1(\bar{K})$.

**Example 8**

Let $\psi(x) = x^2 + c$ for $c \in K$. Then

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Let \( \psi(x) = x^2 + c \) for \( c \in K \). Then

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The pairing $\langle \varphi, \psi \rangle$ can be alternatively defined as the arithmetic intersection product $\mathcal{L}_\varphi \cdot \mathcal{L}_\psi$ between the canonical adelic metrized line bundles $\mathcal{L}_\varphi$ and $\mathcal{L}_\psi$ in the sense of Zhang.

Mimar has proved the equivalence of the following three conditions:

- $\text{PrePer}(\varphi) = \text{PrePer}(\psi)$;
- $\text{PrePer}(\varphi) \cap \text{PrePer}(\psi)$ is infinite;
- $\liminf_{x \in \mathcal{P}_1(\overline{K})} (h_\varphi(x) + h_\psi(x)) = 0$.

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Zhang’s successive minima theorem implies that $\mathcal{L}_\varphi \cdot \mathcal{L}_\psi = 0$ if and only if $\liminf_{x \in \mathbb{P}^1(\bar{K})} (h_\varphi(x) + h_\psi(x)) = 0$. 

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We learned after completing our preprint that our Theorem 1

\[ h_\psi(x_n) \to 0 \quad \Rightarrow \quad h_\varphi(x_n) \to \langle \varphi, \psi \rangle \]

is equivalent to a special case of a recent result of Chambert-Loir and Thuillier.
We learned after completing our preprint that our Theorem 1

\[ h_\psi(x_n) \to 0 \implies h_\varphi(x_n) \to \langle \varphi, \psi \rangle \]

is equivalent to a special case of a recent result of Chambert-Loir and Thuillier.

Kawaguchi-Silverman have studied pairs of rational maps \( \varphi \) and \( \psi \) satisfying \( h_\varphi = h_\psi \). They give a complete classification of such pairs when both \( \varphi \) and \( \psi \) are polynomials, or when at least one of \( \varphi \) and \( \psi \) is a Lattès map associated to an elliptic curve.