Solutions to Selected Homework Problems
Math 9 — Fall 2005

§4.4 Indeterminate Forms and L'Hôpital's Rule

§4.4, #1. (a) Indeterminate, "0/0 form".
(b) Limit exists, "tiny/huge", so $\lim_{x\to a} f(x)/p(x) = 0$.

§4.4, #2. (a) Indeterminate, "$0 \cdot \infty$ form".
(b) Limit has the form "1 times huge", so $\lim_{x\to a} h(x)p(x) = \infty$.

§4.4, #3. (a) Limit has the form "0 minus huge", so

$$\lim_{x\to a} f(x) - p(x) = -\infty.$$

(b) Indeterminate, "$\infty - \infty$ form".

§4.4, #6. Numerator and denominator go to 0, so can use l'Hôpital's rule.

$$\lim_{x\to -2} \frac{x+2}{x^2+3x+2} = \lim_{x\to -2} \frac{1}{2x+3} = \frac{1}{-1} = -1.$$

§4.4, #8. Numerator and denominator go to 0, so can use l'Hôpital's rule.

$$\lim_{x\to 1} \frac{x^a-1}{x^b-1} = \lim_{x\to 1} \frac{ax^{a-1}}{bx^{b-1}} = \frac{a}{b}.$$

§4.4, #11. Numerator and denominator go to 0, so can use l'Hôpital's rule.

$$\lim_{t\to 0} \frac{e^t-1}{t^3} = \lim_{t\to 0} \frac{e^t}{3t^2}.$$

Now the numerator goes to 1 and the denominator goes to 0 (taking only positive values), so the limit is $+\infty$.

§4.4, #12. Numerator and denominator go to 0, so can use l'Hôpital's rule.

$$\lim_{t\to 0} \frac{e^{3t}-1}{t} = \lim_{t\to 0} \frac{3e^{3t}}{1} = 3e^0 = 3.$$

§4.4, #21. Numerator and denominator go to 0, so can use l'Hôpital's rule.

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x\to 0} \frac{e^x - 1}{2x}.$$

The numerator and denominator still go to 0, so can use l'Hôpital's rule again.

$$\lim_{x\to 0} \frac{e^x - 1}{2x} = \lim_{x\to 0} \frac{e^x}{2} = \frac{e^0}{2} = \frac{1}{2}.$$

§4.4, #25. Numerator and denominator go to 0, so can use l'Hôpital's rule.

$$\lim_{x\to 0} \frac{\sin^{-1}(x)}{x} = \lim_{x\to 0} \frac{1/\sqrt{1-x^2}}{1} = 1.$$

§4.4, #34. Numerator and denominator go to ∞, so can use l'Hôpital's rule (though as we'll see, it doesn't help much). Here's what happens when we use l'Hôpital's rule:

$$\lim_{x\to\infty} \frac{\sqrt{x^2+2}}{\sqrt{2x^2+1}} = \lim_{x\to\infty} \frac{x/\sqrt{x^2+2}}{2x/\sqrt{2x^2+1}} = \lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{2\sqrt{x^2+2}}.$$

So that's more or less the same as the original problem! Lesson learned: *Sometimes, even if l'Hôpital's rule can be applied, it doesn't lead to a solution of the problem.*

Instead, we can use the method where we divide top and bottom by an appropriate power of x.

$$\lim_{x\to\infty} \frac{\sqrt{x^2+2}}{\sqrt{2x^2+1}} = \lim_{x\to\infty} \frac{\frac{1}{x}\sqrt{x^2+2}}{\frac{1}{x}\sqrt{2x^2+1}} = \lim_{x\to\infty} \frac{\sqrt{1+\frac{2}{x^2}}}{\sqrt{2+\frac{1}{x^2}}} = \frac{1}{\sqrt{2}}.$$

§4.4, #38. This has the form "huge times huge", that is, it has the form "$\infty \cdot \infty$". So

$$\lim_{x\to\infty} x^2 e^x = \infty.$$

§4.4, #40. This has the form "tiny times huge" (huge in this case means hugely negative), so it is an indeterminate form "$0 \cdot \infty$". We first use some algebra, then l'Hôpital's rule, then....

$$\begin{aligned}
\lim_{x\to 0}(\sin x)(\ln x) &= \lim_{x\to 0}\frac{\ln x}{1/\sin x} && \text{(algebra)}\\
&= \lim_{x\to 0}\frac{1/x}{-\cos x/\sin^2 x} && \text{(l'Hôpital's rule)}\\
&= \lim_{x\to 0} -\frac{\sin^2 x}{x\cos x} && \text{(algebra)}\\
&= \lim_{x\to 0} -\frac{2\sin x\cos x}{-x\sin x+\cos x} && \text{(l'Hôpital's rule)}\\
&= \frac{0}{1}\\
&= 0
\end{aligned}$$

§4.4, #47. This has the form "huge minus huge", so it is an indeterminate form "$\infty - \infty$". We first use some algebra, then l'Hôpital's rule, then....

$$\begin{aligned}
\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right) &= \lim_{x\to\infty} x\left(\sqrt{1+\frac{1}{x}}-1\right) && \text{(algebra, now } \infty\cdot 0 \text{ form)}\\
&= \lim_{x\to\infty}\frac{\sqrt{1+\frac{1}{x}}-1}{1/x} && \text{(algebra, now } \infty/\infty \text{ form)}\\
&= \lim_{x\to\infty}\frac{\frac{1}{2}\left(1+\frac{1}{x}\right)^{-1/2}\left(-x^{-2}\right)}{-x^{-2}} && \text{(l'Hôpital's rule)}\\
&= \lim_{x\to\infty}\frac{1}{2}\left(1+\frac{1}{x}\right)^{-1/2} && \text{(algebra)}\\
&= \frac{1}{2}.
\end{aligned}$$

NOTE: This problem can be done using l'Hôpital's rule, as we just did. But you may prefer to do it using our other method for moving around square roots. Here's a solution using that method.

$$\begin{aligned}
\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right) &= \lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)\cdot\frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x}\\
&= \lim_{x\to\infty}\frac{(x^2+x)-x^2}{\sqrt{x^2+x}+x}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{x\to\infty} \frac{x}{\sqrt{x^2+x}+x} \\
&= \lim_{x\to\infty} \frac{1}{\frac{1}{x}\sqrt{x^2+x}+1} \\
&= \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x}}+1} \\
&= \frac{1}{2}.
\end{aligned}
$$

§4.4, #50. This has the form "huge minus huge", so it is an indeterminate form "$\infty - \infty$". We first use some algebra, then l'Hôpital's rule, then....

$$
\begin{aligned}
\lim_{x\to\infty} (xe^{1/x} - x) &= \lim_{x\to\infty} \frac{e^{1/x}-1}{1/x} && \text{(algebra, now 0/0 form)} \\
&= \lim_{x\to\infty} \frac{e^{1/x}(-1/x^2)}{-1/x^2} && \text{(l'Hôpital's rule)} \\
&= \lim_{x\to\infty} e^{1/x} && \text{(algebra)} \\
&= e^0 = 1
\end{aligned}
$$

§4.4, #53. This has the form "one to the huge power, so it is an indeterminate form "1^∞". We first use take logs, then use some algebra, then l'Hôpital's rule, then....

Let $y = (1-2x)^{1/x}$. Then $\ln y = (1/x)\ln(1-2x)$, so

$$
\begin{aligned}
\lim_{x\to 0} \ln y &= \lim_{x\to 0} \frac{\ln(1-2x)}{x} && \text{(this is a 0/0 form)} \\
&= \lim_{x\to 0} \frac{\frac{1}{1-2x}\cdot(-2)}{1} && \text{(l'Hôpital's rule)} \\
&= \lim_{x\to 0} \frac{-2}{1-2x} && \text{(algebra)} \\
&= -2.
\end{aligned}
$$

Thus $\ln y$ approaches -2, so

$$
\lim_{x\to 0}(1-2x)^{1/x} = \lim_{x\to 0} y = \lim_{x\to 0} e^{\ln y} = e^{-2}.
$$

§4.4, #72. (This was a bonus problem.) Let r be the radius of the circle. The area of the piece of the circle pictured is $\theta/2\pi$ of the entire circle, so it is equal to $(\theta/2\pi)\pi r^2 = \theta r^2/2$. On the other hand, that area is clearly equal to $2B(\theta) + A(\theta)$, so we get the relation

$$2B(\theta) + A(\theta) = \frac{1}{2}\theta r^2.$$

Next we compute the area of the triangle. Its base has length r, and its height has length $r \sin\theta$. So its area is

$$2B(\theta) = \frac{1}{2} r^2 \sin\theta.$$

Dividing the first equation by the second, we find

$$1 + \frac{A(\theta)}{2B(\theta)} = \frac{\frac{1}{2}\theta r^2}{\frac{1}{2} r^2 \sin\theta} = \frac{\theta}{\sin\theta}.$$

Now take the limit

$$1 + \frac{1}{2}\lim_{\theta\to 0}\frac{A(\theta)}{B(\theta)} = \lim_{\theta\to 0}\frac{\theta}{\sin\theta} = \lim_{\theta\to 0}\frac{1}{\cos\theta} = 1.$$

Subtracting 1 from both sides yields

$$\lim_{\theta\to 0}\frac{A(\theta)}{B(\theta)} = 0.$$