Problem 1. (10 points) Compute each of the following limits.

(a) \[
\lim_{x \to -1} \frac{x^3 + 4x^2 - 4x - 7}{x^4 + x^3 + x^2 + x}
\]

(b) \[
\lim_{x \to 0} \frac{\sin^2(x)}{e^x - 1 - x}
\]

Solution. (a) Use L'Hôpital’s rule.

\[
\lim_{x \to -1} \frac{x^3 + 4x^2 - 4x - 7}{x^4 + x^3 + x^2 + x} = \lim_{x \to -1} \frac{3x^2 + 8x - 4}{4x^3 + 3x^2 + 2x + 1}
= \frac{3 - 8 - 4}{-4 + 3 - 2 + 1} = \frac{9}{2}
\]

(b) Use L'Hôpital’s rule (twice).

\[
\lim_{x \to 0} \frac{\sin^2(x)}{e^x - 1 - x} = \lim_{x \to 0} \frac{2\sin(x)\cos(x)}{e^x - 1}
= \lim_{x \to 0} \frac{-2\sin^2(x) + 2\cos^2(x)}{e^x}
= \frac{-2 \cdot 0 + 2 \cdot 1}{e^0}
= 2.
\]

Problem 2. (10 points) (a) Compute the following indefinite integral:

\[
\int \frac{e^x}{e^{2x} + 1} \, dx.
\]

(b) Define a function \( f(x) \) by the formula

\[
f(x) = \int_0^{\sin x} e^t^2 \, dt.
\]

Compute the derivative \( f'(x) \) and its value \( f' (\pi/3) \).
Solution. (a) We make a substitution.

\[
\int \frac{e^x}{e^{2x} + 1} \, dx = \int \frac{1}{u^2 + 1} \, du = \tan^{-1}(u) + C
\]

\(u = e^x, \, du = e^x \, dx\)

= \tan^{-1}(e^x) + C.

(b) We use the fundamental theorem of calculus, together with the chain rule. So if we let \(u(x) = \sin x\), then

\[
f(x) = \int_0^{u(x)} e^{t^2} \, dt,
\]

so

\[
\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = \left( \frac{d}{du} \int_0^u e^{t^2} \, dt \right) \left( \frac{d}{dx} \sin x \right) = e^{u^2} \cos x = e^{\sin^2(x)} \cos(x).
\]

We’re also supposed to evaluate \(f'(x)\) at \(x = \pi/3\). We have \(\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}\) and \(\cos \frac{\pi}{3} = \frac{1}{2}\), so

\[
f'\left(\frac{\pi}{3}\right) = e^{(\sqrt{3}/2)^2} \cdot \frac{1}{2} = \frac{1}{2} e^{3/4}
\]

Problem 3. (10 points) The region in the \(xy\)-plane lying above the curve \(y = x^2\) and below the line \(y = 1\) is rotated about the \(x\)-axis. Compute the volume of the resulting solid.

Solution. The region lies between \(-1 \leq x \leq 1\). The little line segments that are rotated to form washers run from \(y = x^2\) to \(y = 1\). So
the volume is

\[ V = \int_{-1}^{1} \pi \cdot 1^2 - \pi \cdot (x^2)^2 \, dx \]

\[ = \int_{-1}^{1} \pi - \pi \cdot x^4 \, dx \]

\[ = \left( \pi x - \pi \cdot \frac{x^5}{5} \right)_{x=1}^{x=-1} \]

\[ = \left( \pi - \frac{\pi}{5} \right) - \left( -\pi + \frac{\pi}{5} \right) \]

\[ = \frac{8\pi}{5} \]

**Problem 4.** (10 points) For each part, write down a definite integral that computes the given quantity. You DO NOT have to evaluate the integral.

(a) Write down an integral that gives the length of the curve \( y^2 = x^5 \) with \( y > 0 \) and \( 1 \leq x \leq 2 \). (Do not evaluate the integral.)

(b) The curve \( y = \cos x \) with \( 0 \leq x \leq \pi \) is rotated about the \( x \)-axis. Write down an integral that gives the surface area of the resulting surface. (Do not evaluate the integral.)

**Solution.** (a) We have \( y = x^{5/2} \), so \( \frac{dy}{dx} = \frac{5}{2}x^{3/2} \). Then the formula for the arc length of a curve is

\[ L = \int_{1}^{2} \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \]

\[ = \int_{1}^{2} \sqrt{1 + \left( \frac{5}{2}x^{3/2} \right)^2} \, dx \]

\[ = \int_{1}^{2} \sqrt{1 + \frac{25}{4}x^3} \, dx \]

An alternative way to do this problem is to parameterize the curve as

\[ x = t^2, \quad y = t^5, \quad 1 \leq t \leq \sqrt{2}. \]
Then
\[ L = \int_1^{\sqrt{2}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt \]
\[ = \int_1^{\sqrt{2}} \sqrt{(2t^2) + (5t^4)^2} \, dt \]
\[ = \int_1^{\sqrt{2}} \sqrt{4t^2 + 25t^8} \, dt. \]

(b) We have \( \frac{dy}{dx} = -\sin x \), so the formula for the surface area of a surface of revolution gives
\[ A = \int_0^{\pi} 2\pi|y|\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \int_0^{\pi} 2\pi|\cos x|\sqrt{1 + \sin^2 x} \, dx. \]
(Note that \( \cos x \) is sometimes negative on the interval from 0 to \( \pi \), so it’s necessary to use the absolute value sign. Or, by symmetry, just do the integral from 0 to \( \pi/2 \), and then double the result.)

**Problem 5.** (10 points) It takes 12 lbs of force to compress a spring 3 ft, starting from its uncompressed state. How much work is done in compressing the spring 6 ft, starting from its uncompressed state?

**Solution.** The force to compress a spring is given by the formula
\[ F = kx, \]
where \( F \) is the force, \( x \) is the distance compressed, and \( k \) is a constant that depends on the spring. We are told that \( F = 9 \) lbs when \( x = 3 \) ft, so
\[ F(2) = k \cdot 3 = 12, \quad \text{so} \quad k = 4. \]
The unit of force is pounds and the unit of distance is feet, so \( k = 4 \) lb/ft.

The work is given by the integral of the force, so
\[ W = \int_0^6 F(x) \, dx = \int_0^6 4x \, dx = 2x^2 \bigg|_{x=0}^{x=6} = 72 \text{ ft-lb}. \]