

Problem 1. (10 points) Compute each of the following indefinite integrals.

(a) $\int x^3 \ln x \, dx$

(b) $\int \frac{x-1}{x^2+x} \, dx.$

Solution. (a) Use integration by parts. We take

$$u = \ln x, \quad dv = x^3 \, dx, \quad \parallel \quad du = \frac{1}{x} \, dx, \quad v = \frac{1}{4}x^4.$$

So

$$\begin{aligned} \int x^3 \ln x \, dx &= \frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^4 \cdot \frac{1}{x} \, dx \\ &= \frac{1}{4}x^4 \ln x - \frac{1}{4} \int x^3 \, dx \\ &= \boxed{\frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C}. \end{aligned}$$

(b) Use partial fractions.

$$\frac{x-1}{x^2+x} = \frac{A}{x} + \frac{B}{x+1}.$$

$$x-1 = A(x+1) + Bx = (A+B)x + A.$$

So $A+B=1$ and $A=-1$, which gives $B=2$. Now we can integrate

$$\begin{aligned} \int \frac{x-1}{x^2+x} \, dx &= \int \frac{-1}{x} + \frac{2}{x+1} \, dx \\ &= -\ln|x| + 2\ln|x+1| + C \\ &= \boxed{\ln \left| \frac{(x+1)^2}{x} \right| + C}. \end{aligned}$$

Problem 2. (10 points) The following table lists some of the values of a function $f(x)$.

x	0	1	2	3	4	5	6	7	8	9	10
$f(x)$	1	3	2	1	4	3	7	5	4	6	2

Divide the interval from 0 to 8 into 4 subintervals and use Simpson's rule to estimate

$$\int_0^8 f(x) \, dx.$$

Solution. We are using Simpson's rule with $n = 4$. Simpson's rule with $n = 4$ is

$$\int_a^b f(x) dx \approx \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4).$$

In our case $\Delta x = (8 - 0)/4 = 2$, so

$$\begin{aligned} \int_0^8 f(x) dx &\approx \frac{2}{3}(f(0) + 4f(2) + 2f(4) + 4f(6) + f(8)) \\ &= \frac{2}{3}(1 + 4 \cdot 2 + 2 \cdot 4 + 4 \cdot 7 + 4) \\ &= \boxed{\frac{98}{3}}. \end{aligned}$$

Problem 3. (20 points) For each of the following series, state whether it converges or diverges and **give a reason**. To receive credit, you must give a reason for your answer.

(a) $\sum_{n=1}^{\infty} (-1)^n \ln n.$

(b) $\sum_{n=1}^{\infty} \frac{n}{n^2 + n + 1}$

(c) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}.$

(d) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2 + n + 1}$

Solution. (a) The n 'th term of this series does not go to 0 as $n \rightarrow \infty$, so the series **diverges** by the **n 'th term test**. You can see that the n 'th term doesn't go to 0 by observing that

$$\lim_{n \rightarrow \infty} |(-1)^n \ln n| = \infty.$$

(b) We use the limit comparison test. We compare with the series $\sum 1/n$. So first we check that

$$\lim_{n \rightarrow \infty} \frac{\frac{n}{n^2 + n + 1}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2 + n + 1} = 1.$$

(You can compute the limit using L'Hôpital's rule, or by dividing numerator and denominator by n^2 .) Since the limit exists and is positive,

the series

$$\sum_{n=1}^{\infty} \frac{n}{n^2 + n + 1} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n}$$

either both converge or both diverge. But we know that $\sum 1/n$ diverges from class, it is a p -series with $p = 1$. (Or you can use the integral test and note that $\int_1^{\infty} dt/t$ diverges.) Hence the series

$$\sum_{n=1}^{\infty} \frac{n}{n^2 + n + 1} \quad \boxed{\text{diverges by the limit comparison test}}.$$

(c) We use the integral test. Thus

$$\begin{aligned} \int \frac{dx}{x(\ln x)^2} &= \int \frac{du}{u^2} && \boxed{u = \ln x, \quad du = \frac{dx}{x}} \\ &= -\frac{1}{u} \\ &= -\frac{1}{\ln x}. \end{aligned}$$

So the improper integral

$$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{b \rightarrow \infty} \int_2^b \frac{dx}{x(\ln x)^2} = \lim_{b \rightarrow \infty} -\frac{1}{\ln x} \Big|_{x=2}^b = \lim_{b \rightarrow \infty} -\frac{1}{\ln b} + \frac{1}{2} = \frac{1}{2}$$

converges. Hence the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} \quad \boxed{\text{converges by the integral test}}.$$

(d) The sequence of values

$$\frac{n}{n^2 + n + 1} \quad \text{for } n = 1, 2, 3, \dots$$

is positive, decreasing, and goes to 0 as $n \rightarrow \infty$, so the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2 + n + 1} \quad \boxed{\text{converges by the alternating series test}}.$$

Problem 4. (10 points) Describe the set of values of x for which the following series converges. Be sure to explain why your answer is correct.

$$\sum_{n=1}^{\infty} \frac{x^n}{n}.$$

Solution. (a) We use the ratio test to find the radius of convergence. So we first compute

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}} \right| = \lim_{n \rightarrow \infty} \left| x \cdot \frac{n}{n+1} \right| = |x|.$$

So the series converges for all $|x| < 1$ and diverges for all $|x| > 1$. We also need to check what happens when $x = 1$ and $x = -1$. For $x = 1$ we get the harmonic series $\sum 1/n$, which we know diverges, for example by the integral test. For $x = -1$, we get the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots.$$

The sequence $1/n$ is positive, decreasing, and goes to 0, so the series converges by the alternating series test. So we find that the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n} \quad \boxed{\text{converges for } -1 \leq x < 1, \text{ and diverges for all other values of } x.}$$

Problem 5. (10 points) (a) The following table gives the values of a function $f(x)$ and some of its derivatives at various points. Write down the Taylor polynomial of degree 3 for $f(x)$ at the point $x = 2$.

$f(0) = 1$	$f'(0) = 3$	$f''(0) = 2$	$f'''(0) = 5$
$f(1) = 2$	$f'(1) = 7$	$f''(1) = -3$	$f'''(1) = 4$
$f(2) = 3$	$f'(2) = 5$	$f''(2) = -7$	$f'''(2) = 4$

(b) The Taylor series of the function $g(x)$ at $x = 0$ is the series

$$g(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots = \sum_{n=0}^{\infty} (n+1)x^n.$$

Write down the Taylor series of $g(\frac{1}{2}x)$ at $x = 0$ and the Taylor series of $g'(x)$ at $x = 0$.

Solution. (a) The Taylor polynomial of degree 3 consists of the first four terms of the Taylor series. In general at $x = a$ this is the polynomial

$$f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{6}f'''(a)(x-a)^3.$$

The problem asks for the situation that $a = 2$. We use the values

$$f(2) = 3, \quad f'(2) = 5, \quad f''(2) = -7, \quad f'''(2) = 4.$$

This gives the answer

$$\begin{aligned} f(2) + f'(2)(x-2) + \frac{1}{2}f''(2)(x-2)^2 + \frac{1}{6}f'''(2)(x-2)^3 \\ = \boxed{3 + 5(x-2) - \frac{7}{2}(x-2)^2 + \frac{2}{3}(x-2)^3}. \end{aligned}$$

(b) The series for $g(\frac{1}{2}x)$ is obtained by taking the series for $g(x)$ and replacing x with $\frac{1}{2}x$. So

$$g\left(\frac{1}{2}x\right) = 1 + x + \frac{3}{4}x^2 + \frac{1}{2}x^3 + \frac{5}{16}x^4 + \cdots = \boxed{\sum_{n=0}^{\infty} \frac{(n+1)x^n}{2^n}}.$$

The series for $g'(x)$ is obtained by differentiating each term of the series for $g(x)$, so

$$g'(x) = 2 + 6x + 12x^2 + 20x^3 + \cdots = \sum_{n=0}^{\infty} (n+1)nx^{n-1}.$$

This series isn't quite in the correct form, since when $n = 0$, we get a term with x^{-1} . Of course, that term disappears since it is multiplied by n , which is 0. But it's best to re-index the series. So

$$g'(x) = 2 + 6x + 12x^2 + 20x^3 + \cdots = \boxed{\sum_{n=0}^{\infty} (n+2)(n+1)x^n}.$$