Problem 1. (10 points)
(a) The area above the x-axis and under the curve $y = e^x + 1$ with $0 \le x \le 1$ is rotated about the x-axis. What is the volume of the resulting solid?

(b) The curve $y = e^x$ with $0 \le x \le 1$ is rotated about the line $y = -1$. Write down a definite integral that gives the **surface area** of the resulting surface. NOTE: You **do not** need to evaluate the integral in part (b).

Solution. (a) The volume is given by the formula

$$\begin{aligned} V &= \int_a^b \pi f(x)^2 \, dx \\ &= \int_0^1 \pi (e^x + 1)^2 \, dx \\ &= \pi \int_0^1 e^{2x} + 2e^x + 1 \, dx \\ &= \pi \left(\frac{1}{2} e^{2x} + 2e^x + x \right) \Big|_{x=0}^{x=1} \\ &= \pi \left(\frac{1}{2} e^2 + 2e + 1 \right) - \pi \left(\frac{1}{2} + 2 \right) \\ &= \boxed{\pi \left(\frac{1}{2} e^2 + 2e - \frac{3}{2} \right)} \end{aligned}$$

(b) The surface area is (note that the line of rotation is $y = -1$, not the x-axis)

$$\begin{aligned} A &= \int_a^b 2\pi (y+1) \sqrt{1 + \left(\frac{dy}{dx} \right)^2} \, dx \\ &= \boxed{\int_0^1 2\pi (e^x + 1) \sqrt{1 + e^{2x}} \, dx} \end{aligned}$$

Problem 2. (10 points)
(a) Compute the value of the following limit:

$$\lim_{x \to 0} \frac{\sin(x) - x\cos(x)}{x^3}.$$

(b) Find all values of x that make the following equation true?

$$2 + 2x^2 + 2x^4 + 2x^6 + 2x^8 + \cdots = \frac{9}{4}.$$

Solution. (a) There are two ways to do this problem. First is to use L'Hôpital's rule (more than once):

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin(x) - x\cos(x)}{x^3} &= \lim_{x\to 0} \frac{\cos(x) - \cos(x) + x\sin(x)}{3x^2} && \text{(L'Hôpital)} \\
&= \lim_{x\to 0} \frac{x\sin(x)}{3x^2} && \text{(algebra)} \\
&= \lim_{x\to 0} \frac{\sin(x)}{3x} && \text{(algebra)} \\
&= \lim_{x\to 0} \frac{\cos(x)}{3} && \text{(L'Hôpital)} \\
&= \boxed{\frac{1}{3}}.
\end{aligned}$$

The second way is to use the Taylor series expansions of sin and cos. Thus

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin(x) - x\cos(x)}{x^3} &= \lim_{x\to 0} \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) - x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)}{x^3} \\
&= \lim_{x\to 0} \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) - \left(x - \frac{x^3}{2!} + \frac{x^5}{4!} - \cdots\right)}{x^3} \\
&= \lim_{x\to 0} \frac{-\frac{x^3}{3!} + \frac{x^3}{2!} + \cdots}{x^3} \\
&= -\frac{1}{6} + \frac{1}{2} \\
&= \boxed{\frac{1}{3}}.
\end{aligned}$$

(b) The series

$$2 + 2x^2 + 2x^4 + 2x^6 + 2x^8 + \cdots$$

is a geometric series $a + ar + ar^2 + ar^3 + \cdots$ with $a = 2$ and $r = x^2$. The value of the geometric series is

$$a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r} \quad \text{for } |r| < 1,$$

so

$$2 + 2x^2 + 2x^4 + 2x^6 + 2x^8 + \cdots = \frac{2}{1 - x^2}.$$

So we set

$$\frac{2}{1 - x^2} = \frac{9}{4}$$

and solve for x. This gives

$$\frac{8}{9} = 1 - x^2$$
$$x^2 = \frac{1}{9}$$
$$x = \boxed{\pm\frac{1}{3}}.$$

Problem 3. (10 points)

(a) Compute the indefinite integral

$$\int x^2 e^{-x}\, dx$$

(b) Compute the indefinite integral

$$\int \cos(\sqrt{x})\, dx.$$

Solution. (a) We integrate by parts (twice). First use

$$u = x^2, \quad du = 2x\, dx, \quad dv = e^{-x}\, dx, \quad v = -e^{-x},$$

to get

$$\int x^2 e^{-x}\, dx = -x^2 e^{-x} + \int e^{-x} 2x\, dx.$$

Next use

$$u = 2x, \quad du = 2\, dx, \quad dv = e^{-x}\, dx, \quad v = -e^{-x},$$

on the right-hand integral to get

$$\int x^2 e^{-x}\, dx = -x^2 e^{-x} + \left(-2xe^{-x} - \int e^{-x} 2\, dx \right)$$
$$= \boxed{-x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + C}.$$

(b) We make the substitution $u = \sqrt{x}$, which is $u^2 = x$, so $2u\,du = dx$. This gives

$$\int \cos(\sqrt{x})\,dx = \int \cos(u)\,2u\,du.$$

We now need to integrate by parts, but it will be very confusing using u for two different things, to we replace u in the integral by some other letter, say z. (So $z = \sqrt{x}$.) We want to compute

$$\int 2z\cos(z)\,dz.$$

We take

$$u = 2z, \qquad du = 2dz, \qquad dv = \cos(z)\,dz, \qquad v = \sin(z).$$

This gives

$$\int 2z\cos(z)\,dz = 2z\sin(z) - \int \sin(z)\,2dz = 2z\sin(z) + 2\cos(z) + C.$$

Finally, using $z = \sqrt{x}$, we get the solution,

$$\int \cos(\sqrt{x})\,dx = \boxed{2\sqrt{x}\sin(\sqrt{x}) + 2\cos(\sqrt{x}) + C.}$$

Problem 4. (10 points)

(a) Compute the value of the improper integral

$$\int_0^\infty xe^{-\pi x^2}\,dx$$

(b) Compute the value of the improper integral

$$\int_2^\infty \frac{1}{x(\ln x)^3}\,dx.$$

Solution. (a) We make the substitution $u = x^2$, so $du = 2x\,dx$. This gives

$$\int_0^\infty xe^{-\pi x^2}\,dx = \int_0^\infty e^{-\pi u}\,\frac{du}{2} = \frac{1}{2}\cdot\frac{e^{-\pi u}}{-\pi}\Big|_{u=0}^{u=\infty} = 0 - \frac{1}{2}\cdot\frac{e^0}{-\pi} = \boxed{\frac{1}{2\pi}}$$

(b) We make the substitution $u = \ln x$ and $du = dx/x$ to get

$$\int \frac{1}{x(\ln x)^3}\,dx = \int \frac{1}{u^3}\,du = -\frac{1}{2u^2} + C = -\frac{1}{2(\ln x)^2} + C.$$

Then

$$\begin{aligned}
\int_2^\infty \frac{1}{x(\ln x)^3}\,dx &= \lim_{b\to\infty}\int_2^b \frac{1}{x(\ln x)^3}\,dx \\
&= \lim_{b\to\infty} -\frac{1}{2(\ln x)^2}\bigg|_{x=2}^{x=b} \\
&= \lim_{b\to\infty} -\frac{1}{2(\ln b)^2} + \frac{1}{2(\ln 2)^2} \\
&= \boxed{\frac{1}{2(\ln 2)^2}}.
\end{aligned}$$

Problem 5. (10 points) Here are a few integral formulas taken from the table of integrals in the back of the book.

$$\textbf{[20]} \qquad \int \frac{dx}{\sqrt{a^2+x^2}} = \ln\left(x+\sqrt{a^2+x^2}\right)+C$$

$$\textbf{[22]} \qquad \int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\ln\left(x+\sqrt{a^2+x^2}\right)+C$$

$$\textbf{[23]} \qquad \int \frac{\sqrt{a^2+x^2}}{x}\,dx = \sqrt{a^2+x^2} - a\ln\left|\frac{a+\sqrt{a^2+x^2}}{x}\right|+C$$

Use one or more of these formulas to compute the following integral:

$$\int \sqrt{1+4x^2}\,dx$$

Solution. The integral we want to do is almost in the right form to use formula [22]. There are two easy ways to put it into the right form. First, we can factor a 4 out of the square root, so

$$\int \sqrt{1+4x^2}\,dx = 2\int\sqrt{\frac{1}{4}+x^2}\,dx.$$

Now we can use [22] with $a=\frac{1}{2}$, remembering to multiply the final answer by 2. Thus

$$\begin{aligned}
\int \sqrt{1+4x^2}\,dx &= 2\left(\frac{x}{2}\sqrt{\frac{1}{4}+x^2}+\frac{1}{8}\ln\left(x+\sqrt{\frac{1}{4}+x^2}\right)\right)+C \\
&= \boxed{x\sqrt{\frac{1}{4}+x^2}+\frac{1}{4}\ln\left(x+\sqrt{\frac{1}{4}+x^2}\right)+C}
\end{aligned}$$

As an alternative, one might move the factor of 2 under the square root signs, which gives the equivalent answer

$$\int \sqrt{1+4x^2}\,dx = 2\left(\frac{x}{2}\sqrt{\frac{1}{4}+x^2}+\frac{1}{8}\ln\left(x+\sqrt{\frac{1}{4}+x^2}\right)\right)+C$$

$$= \boxed{\frac{x}{2}\sqrt{1+4x^2}+\frac{1}{8}\ln\left(x+\sqrt{1+4x^2}\right)+C}$$

The other way to put the integral into a form that allow us to use the table is to substitute $u = 2x$, so $du = 2\,dx$. Then

$$\int \sqrt{1+4x^2}\,dx = \int \sqrt{1+u^2}\,\frac{du}{2} = \frac{1}{2}\int \sqrt{1+u^2}\,du.$$

Now [22] gives the value of the integral, and substituting $u = 2x$ gives the final answer.

Problem 6. (10 points) For each of the following series, state whether it converges or diverges and **give a reason**. To receive credit, you must give a reason for your answer.

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n}{n+\sqrt{n}}$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n^2+\sqrt{n}}$

Solution. (a) The n'th term of this series does not go to 0 as $n \to \infty$, so the series $\boxed{\text{diverges}}$ by the $\boxed{n\text{'th term test}}$. You can see that the n'th term doesn't go to 0 by observing that

$$\lim_{n\to\infty}\left|(-1)^n\frac{n}{n+\sqrt{n}}\right| = 1.$$

(b) We use the limit comparison test. We compare with the series $\sum 1/n$. So first we check that

$$\lim_{n\to\infty}\frac{\dfrac{n}{n^2+\sqrt{n}}}{\dfrac{1}{n}} = \lim_{n\to\infty}\frac{n^2}{n^2+\sqrt{n}} = 1.$$

(You can compute the limit using L'Hôpital's rule, or by dividing numerator and denominator by n^2.) Since the limit exists and is positive, the series

$$\sum_{n=1}^{\infty}\frac{n}{n^2+\sqrt{n}} \qquad \text{and} \qquad \sum_{n=1}^{\infty}\frac{1}{n}$$

either both converge or both diverge. But we know that $\sum 1/n$ diverges from class, it is a p-series with $p = 1$. (Or you can use the integral test and note that $\int_1^\infty dt/t$ diverges.) Hence the series

$$\sum_{n=1}^{\infty} \frac{n}{n^2+\sqrt{n}} \quad \boxed{\text{diverges by the limit comparison test}}.$$

Problem 7. (10 points) For each of the following series, state whether it converges or diverges and **give a reason**. To receive credit, you must give a reason for your answer.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3+\sqrt{n}}$

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+\sqrt{n}}$

Solution. (a) We use the limit comparison test. We compare with the series $\sum 1/n^2$. So first we check that

$$\lim_{n\to\infty} \frac{\dfrac{n}{n^2+\sqrt{n}}}{\dfrac{1}{n^2}} = \lim_{n\to\infty} \frac{n^3}{n^3+\sqrt{n}} = 1.$$

(You can compute the limit using L'Hôpital's rule, or by dividing numerator and denominator by n^3.) Since the limit exists and is positive, the series

$$\sum_{n=1}^{\infty} \frac{n}{n^3+\sqrt{n}} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{n^2}$$

either both converge or both diverge. But we know that $\sum 1/n^2$ diverges from class, it is a p-series with $p = 2$. (Or you can use the integral test and note that $\int_1^\infty dt/t^2$ converges.) Hence the series

$$\sum_{n=1}^{\infty} \frac{n}{n^3+\sqrt{n}} \quad \boxed{\text{converges by the limit comparison test}}.$$

(b) The sequence of values

$$\frac{n}{n^2+\sqrt{n}} \quad \text{for } n = 1, 2, 3, \ldots$$

is positive, decreasing, and goes to 0 as $n \to \infty$, so the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+\sqrt{n}} \quad \boxed{\text{converges by the alternating series test}}.$$

Problem 8. (10 points) Let $f(x)$ be the function

$$f(x) = \int_0^x \frac{\sin(t)}{t}\,dt$$

(a) Find the Taylor series expansion of $f(x)$ around $a = 0$. (Either write the answer using summation notation, or write out the first four non-zero terms so that the pattern is clear.)

(b) Compute the derivative $f'(x)$. (*Hint*: For this part, don't use series. In particular, your answer should not be a series.)

Solution. (a) We start with the series expansion

$$\frac{\sin t}{t} = \frac{1}{t}\sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n+1)!}.$$

Integrating, we get

$$\int_0^x \frac{\sin(t)}{t}\,dt = \int_0^x \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n+1)!}\,dx$$
$$= \boxed{\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)(2n+1)!}}$$

The first four non-zero terms are

$$\boxed{x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \frac{x^7}{7\cdot 7!} + \cdots}$$

(b) The (second) fundamental theorem of calculus tells us that

$$f'(x) = \boxed{\frac{\sin(x)}{x}}.$$

Problem 9. (10 points) For what values of x does the following power series converge. Be sure to explain why your answer is correct.

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2}.$$

Solution. We start with the ratio test:

$$\rho = \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}/(n+1)^2}{(x-2)^n/n^2}\right| = \lim_{n\to\infty}\left|\frac{n^2}{(n+1)^2}(x-2)\right| = |x-2|.$$

So the series is absolutely convergent for $|x-2|<1$. This is the interval $1<x<3$. Also the power series diverges for $|x-2|>1$ by the n'th term test, since if $|x-2|>1$, then the limit

$$\lim_{n\to\infty}\frac{(x-2)^n}{n^2}$$

does not exist.

Next we check the endpoints. When $x=1$ we get the series

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2},$$

and when $x=3$ we get the series

$$\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

The latter converges since it's a p-series with $p=2$, and the former converges absolutely, since

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n^2}\right|=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

is also a convergent p-series. So the power series converges for $x=1$ and $x=3$, which means that it converges on the interval

$$\boxed{1\le x\le 3}$$

and it diverges for all other values of x.

Problem 10. (10 points)

(a) Find the general solution to the differential equation

$$\frac{dy}{dx}+3x^2y=2x^2.$$

(b) Find the particular solution to the differential equation in (a) that satisfies the initial condition $y(0)=5$.

Solution. (a) This is a first order linear differential equation. The integrating factor is

$$v(x)=e^{\int 3x^2\,dx}=e^{x^3}.$$

Multiplying by $v(x)$, we get

$$\frac{d}{dx}\left(e^{x^3}y\right)=2x^2e^{x^3}.$$

Integrating both sides gives

$$e^{x^3}y = \int 2x^2e^{x^3}\,dx = \frac{2}{3}e^{x^3} + C.$$

(To do the integral, make the substitution $u = x^3$ and $du = 3x^2\,dx$.) Dividing by e^{x^3} gives the general solution

$$\boxed{y(x) = \frac{2}{3} + \frac{C}{e^{x^3}}}$$

(b) We want $y(0) = 5$, so

$$5 = y(0) = \frac{2}{3} + C.$$

This gives $C = 13/3$, so the specific solution we want is

$$\boxed{y(x) = \frac{2}{3} + \frac{13}{3e^{x^3}}}$$

Problem 11. (10 points)

(a) Find the general solution to the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 3y = 0$$

(b) Suppose that the differential equation

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

has the two solutions

$$y_1(x) = e^{-3x} \quad \text{and} \quad y_2(x) = xe^{-3x}.$$

What are the values of b and c?

Solution. (a) The auxilliary equation is

$$r^2 - 2r + 3 = 0.$$

Using the quadratic formula gives

$$r = \frac{2 \pm \sqrt{4-12}}{2} = \frac{2 \pm \sqrt{-8}}{2} = 1 + i\sqrt{2}.$$

So the general solution is

$$\boxed{y(x) = e^x\big(c_1\cos(\sqrt{2}\,x) + c_2\sin(\sqrt{2}\,x)\big)}$$

(b) The form of the solutions says that the auxilliary equations must have $r = -3$ as its only root, so it must be $(r+3)^2$. So

$$r^2 + br + c = (r+3)^2 = r^2 + 6r + 9.$$

This gives the values $\boxed{b = 6 \text{ and } c = 9}$.

Problem 12. (10 points) A 40-gallon tank is half full of brine containing 100 pounds of salt. Brine with a concentration of 10 pounds of salt per gallon of water is pumped into the tank at the rate of 6 gallons per minute, while the well-stirred mixture is drained out at the rate of 4 gallons per minute.

(a) How many minutes does it take until the tank is full of water?

(b) Find a formula for the amount of salt in the tank after t minutes.

(c) How much salt is in the tank when the tank is full of water?

Solution. We start by setting notation:

$$\begin{aligned} t &= \text{time, measured in minutes,} \\ V(t) &= \text{water in tank, measured in gallons,} \\ y(t) &= \text{salt in bucket, measured in lbs.} \end{aligned}$$

We are given that

$$V(0) = 20 \text{ gallons and } y(0) = 100 \text{ lbs.}$$

(The tank holds 40 gallons total, and it is half full.) We have

$$\begin{aligned} \frac{dV}{dt} &= -(\text{water rate in}) - (\text{water rate out}) \\ &= 6 - 4 \\ &= 2 \text{ gallons/minute.} \end{aligned}$$

Integrating gives $V(t) = 2t + C$, and using $V(0) = 20$ gives $C = 20$, so

$$V(t) = 2t + 20.$$

The tank holds 40 gallons, so it is full when $V(t) = 40$, which is when $t = 10$. So the answer to (a) is:

$$\boxed{\text{The tank is full of water after 10 minutes.}}$$

Next we compute

$$\frac{dy}{dt} = (\text{rate salt flows in}) - (\text{rate salt flows out}).$$

The first quantity is given by

$$\begin{aligned}(\text{rate salt flows in}) &= \frac{(10 \text{ lbs salt})}{(1 \text{ gallon water})} \cdot \frac{(6 \text{ gallons water})}{(1 \text{ minute})} \\ &= 60 \text{ lbs/min.}\end{aligned}$$

The second quantity is

$$\begin{aligned}&(\text{rate salt flows out}) \\ &\qquad = (\text{rate water flows out}) \cdot (\text{concentration of salt in water}) \\ &\qquad = (\text{rate water flows out}) \cdot \frac{(\text{amount of salt in water})}{(\text{amount of water})} \\ &\qquad = 4 \cdot \frac{y(t)}{V(t)} \\ &\qquad = 4 \cdot \frac{y(t)}{2t+20}\end{aligned}$$

So we find that

$$\frac{dy}{dt} = 60 - 4\frac{y}{2t+20}.$$

This gives the linear differential equation

$$\frac{dy}{dt} + 4\frac{y}{2t+20} = 60.$$

The integrating factor is

$$e^{\int 4/(2t+20)\, dt} = e^{2\ln(2t+20)} = (2t+20)^2.$$

Multiplying by the integrating factor gives

$$\begin{aligned}(2t+20)^2\frac{dy}{dt} + 4(2t+20)y &= 60(2t+20)^2, \\ \frac{d}{dt}\big((2t+20)^2 y\big) &= 60(2t+20)^2, \\ (2t+20)^2 y &= \int 60(2t+20)^2\, dt, \\ (2t+20)^2 y &= 10(2t+20)^3 + C, \\ y &= 10(2t+20) + \frac{C}{(2t+20)^2}.\end{aligned}$$

To find C, we use the initial condition $y(0) = 100$, so

$$100 = y(0) = 10 \cdot 20 + \frac{C}{20^2} = 200 + \frac{C}{400}.$$

This gives $C = 40,000$, and we have arrived at the answer to (b):

$$\boxed{y(t) = 20t + 200 - \frac{40000}{(2t+20)^2}.}$$

The answer to (c) is now easy. The tank is full of water after $t = 10$ minutes, so the amount of salt is

$$y(10) = 200 + 200 - \frac{40000}{40^2} = \boxed{375 \text{ lbs of salt}}.$$

Problem 13. (10 points) The non-homogeneous differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - y = 3xe^{2x}$$

has a particular solution of the form

$$y(x) = Axe^{2x} + Be^{2x}.$$

What are the values of A and B?

Solution. We first compute

$$\begin{aligned} y(x) &= Axe^{2x} + Be^{2x}, \\ y'(x) &= Ae^{2x} + 2Axe^{2x} + 2Be^{2x} \\ &= 2Axe^{2x} + (A+2B)e^{2x}, \\ y''(x) &= 2Ae^{2x} + 4Axe^{2x} + 2(A+2B)e^{2x} \\ &= 4Axe^{2x} + (4A+4B)e^{2x}. \end{aligned}$$

Substituting into the differential equation gives

$$\begin{aligned} y'' + 2y' - y &= \big(4Axe^{2x} + (4A+4B)e^{2x}\big) + 2\big(2Axe^{2x} + (A+2B)e^{2x} \\ &\qquad - \big(Axe^{2x} + Be^{2x}\big) \\ &= 7Axe^{2x} + (6A+7B)e^{2x}. \end{aligned}$$

We want this to equal $3xe^{2x}$, so we need

$$7A = 3 \qquad \text{and} \qquad 6A + 7B = 0.$$

The first equation gives $A = 3/7$, and then the second equation gives $B = -6A/7 = -18/49$. So

$$\boxed{A = \frac{3}{7} \quad \text{and} \quad B = -\frac{18}{49}.}$$

Problem 14. (10 points) The differential equation

$$\frac{d^2y}{dx^2} - x\frac{dy}{dx} - xy = 0$$

has a series solution whose first few terms are

$$y(x) = 2 + 5x + c_2x^2 + c_3x^3 + c_4x^4 + \dots.$$

What are the values of c_2, c_3, and c_4?

Solution. We write

$$\begin{aligned} y &= 2 + 5x + c_2x^2 + c_3x^3 + \cdots, \\ y' &= 5 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \cdots, \\ y'' &= c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + \cdots. \end{aligned}$$

So this gives

$$\begin{aligned} -xy &= \quad - 2x - 5x^2 - c_2x^3 - \cdots, \\ -xy' &= \quad - 5x - 2c_2x^2 - 3c_3x^3 - \cdots, \\ y'' &= c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + \cdots. \end{aligned}$$

Adding them gives

$$y''-xy'-xy = c_2+(6c_3-7)x+(12c_4-2c_2-5)x^2+(20c_5-3c_3-c_2)x^3+\cdots.$$

This is supposed to equal zero, so in particular

$$c_2 = 0, \qquad 6c_3 - 7 = 0, \qquad \text{and} \qquad 12c_4 - 2c_2 - 5 = 0.$$

This gives

$$\boxed{c_2 = 0, \qquad c_3 = \frac{7}{6}, \qquad \text{and} \qquad c_4 = \frac{5}{12}}$$

Problem 15. (10 points) An 8-lb weight stretches a spring 17 feet. The spring-mass system resides in a medium offering a resistance to the motion that is numerically equal to 6 times the instanteneous velocity. The weight is released at a position 4 feet below the equilibrium position with a downward velocity of 9 ft/sec.

Write down a **differential equation** and **initial conditions** that model the given situation. (You may use the fact that the gravitational constant is $g = 32$ ft/sec^2.)

NOTE: You **do not** have to solve the differential equation.

Solution. We can find the spring constant using

$$F = ks \quad \text{with } F = 8 \text{ lb and } s = 17 \text{ ft, so } k = \frac{8}{17} \text{ lb/ft.}$$

The differential equation of motion is

$$my'' + \delta y' + ky = 0,$$

where the mass is

$$m = \frac{8 \text{ lb}}{32 \text{ ft/sec}^2} = \frac{1}{4},$$

the friction constant is given as $\delta = 6$, and we found the spring constant earlier to be $k = \frac{8}{17}$. So the differential equation describing the motion is

$$\boxed{\frac{1}{4}y'' + 6y' + \frac{8}{17}y = 0}$$

The initial conditions are

$$\boxed{y(0) = 17 \text{ ft} \quad \text{and} \quad y'(0) = 9 \text{ ft/sec}}$$

Problem 16. (10 points) The polar coordinate equation

$$r = 1 + \cos\theta, \qquad 0 \leq \theta \leq 2\pi,$$

describes a cardioid. Find the area inside the part of this cardioid that lies in the first quadrant, i.e., in the region with $x \geq 0$ and $y \geq 0$.

Solution. The quantity $1 + \cos\theta$ is never negative, so when we graph $r = 1+\cos\theta$, the value of r is never negative. So a point of the cardioid is in the first quadrant exactly when $0 \leq \theta \leq \frac{1}{2}\pi$. Thus we want the part inside the cardioid with θ in this range.

The formula for area using polar coordinates is

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta.$$

So in our case,

$$\begin{aligned}
A &= \int_0^{\pi/2} \frac{1}{2}(1+\cos\theta)^2\,d\theta \\
&= \frac{1}{2}\int_0^{\pi/2} 1 + 2\cos\theta + \cos^2\theta\,d\theta \\
&= \frac{1}{2}\int_0^{\pi/2} 1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\,d\theta \\
&= \frac{1}{2}\left(\theta + 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\Bigg|_{\theta=0}^{\theta=\pi/2} \\
&= \frac{1}{2}\left(\frac{\pi}{2} + 2 + \frac{\pi}{4}\right) \\
&= \boxed{\frac{3\pi}{8} + 1}
\end{aligned}$$