

Math 100 Review Sheet

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This outline of Math 100 is a summary of the material covered in the course. It is designed to be a study aid, but it is only an outline and should be used as an *addition* to your notes, homework assignments, and the book.

1 Derivatives and Integrals

1.1 L'Hôpital's Rule

If $\lim_{x \rightarrow a} f(x) = 0$ and $\lim_{x \rightarrow a} g(x) = 0$, then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$.

Same if $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = \infty$.

1.2 Definite Integrals

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x,$$

where $\Delta x = \frac{b-a}{n}$, where the interval $a \leq x \leq b$ is divided into n pieces of width Δx , and where x_i is in the i 'th piece.

1.3 Fundamental Theorems of Calculus

Part I: $\int_a^b f(x) dx = F(b) - F(a)$, where $F(x)$ is an antiderivative of $f(x)$.

Part II: Define $F(x) = \int_a^x f(t) dt$. Then $F'(x) = f(x)$.

2 Applications of Definite Integrals

2.1 Volumes by Slicing

$$V = \int_a^b A(x) dx, \quad \text{where } A(x) = \text{cross-sectional area.}$$

2.2 Volumes of Revolution

Rotate region under $y = f(x)$ with $a \leq x \leq b$ around x -axis.

$$V = \int_a^b \pi f(x)^2 dx.$$

2.3 Length of a Curve

Length of $y = f(x)$ for $a \leq x \leq b$.

$$L = \int_a^b \sqrt{1 + f'(x)^2} dx.$$

2.4 Surface Area

Rotate curve $y = f(x)$ for $a \leq x \leq b$ around x -axis.

$$A = \int_a^b 2\pi f(x) \sqrt{1 + f'(x)^2} dx.$$

3 Methods of Integration

3.1 Substitution

Substitute $x = g(u)$ to get

$$\int f(x) dx = \int f(g(u))g'(u) du.$$

3.2 Integration by Parts

$$\int u dv = uv - \int v du.$$

3.3 Partial Fractions

Example: Set

$$\frac{x^2 + 3x - 2}{x^2(x-1)(x^2+x+2)} \text{ equal to } \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{Dx+E}{x^2+x+2}.$$

Clear denominators, equate powers of x , solve for A, B, C, D, E .

3.4 Integral Tables

Manipulate given integral to make it match an integral in the table.

3.5 Numerical Integration

Trapazoidal Rule

$$\int_a^b f(x) dx \approx \frac{\Delta x}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n).$$

Simpson's Rule

$$\int_a^b f(x) dx \approx \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n).$$

3.6 Improper Integrals

$$\int_a^\infty f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx,$$

and similarly for $\int_{-\infty}^b f(x)$, and similarly for $\int_a^b f(x) dx$ if $f(x)$ is not defined at some point in the interval.

Can check convergence or divergence by comparison.

If $0 \leq f(x) \leq g(x)$ for all x and if $\int_a^\infty g(x) dx$ converges,
then $\int_a^\infty f(x) dx$ also converges.

If $0 \leq g(x) \leq f(x)$ for all x and if $\int_a^\infty g(x) dx$ diverges,
then $\int_a^\infty f(x) dx$ also diverges.

4 Sequences and Series

4.1 Sequences

A sequence is a list of numbers a_1, a_2, a_3, \dots

4.2 Series

The infinite series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$$

is the limiting value of the sequence of partial sums

$$s_1 = a_1, \quad s_2 = a_1 + a_2, \quad s_3 = a_1 + a_2 + a_3, \dots$$

4.3 Some Special Series

4.3.1 Geometric Series

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^2 + \dots = \begin{cases} \frac{a}{1-r} & \text{if } |r| < 1, \\ \text{diverges} & \text{if } |r| \geq 1. \end{cases}$$

4.3.2 p -series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots \quad \begin{cases} \text{converges} & \text{if } p > 1, \\ \text{diverges} & \text{if } p \leq 1. \end{cases}$$

(The p -series with $p = 1$ is called the harmonic series.)

4.4 Convergence Tests

4.4.1 n 'th Term Test

If $\lim_{n \rightarrow \infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

4.4.2 Integral Test

$a_n = f(n)$ with $f(x) \geq 0$. Then

$\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x) dx$ either both converge or both diverge.

4.5 Comparison Tests

If $0 \leq a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

If $0 \leq b_n \leq a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

4.6 Ratio Test

For $a_n \geq 0$, compute

$$\rho = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}.$$

Then

$$\sum_{n=1}^{\infty} a_n \begin{cases} \text{converges if } \rho < 1, \\ \text{diverges if } \rho > 1, \\ \text{test is inconclusive if } \rho = 1. \end{cases}$$

4.7 Root Test

For $a_n \geq 0$, compute

$$\rho = \lim_{n \rightarrow \infty} \sqrt[n]{a_n}.$$

Then

$$\sum_{n=1}^{\infty} a_n \begin{cases} \text{converges if } \rho < 1, \\ \text{diverges if } \rho > 1, \\ \text{test is inconclusive if } \rho = 1. \end{cases}$$

4.8 Alternating Series Test

If

$$u_n \geq 0, \quad \text{and} \quad u_n \geq u_{n+1}, \quad \text{and} \quad \lim_{n \rightarrow \infty} u_n = 0,$$

then

$$\sum_{n=1}^{\infty} (-1)^{n-1} u_n = u_1 - u_2 + u_3 - u_4 + \cdots \quad \text{converges.}$$

5 Power Series

5.1 Power Series

A power series is a series of the form

$$\sum_{n=0}^{\infty} a_n (x - a)^n.$$

It will generally converge for all x in some interval centered at a , which may or may not include the endpoints. The ratio and root tests are good for finding the interval, then use other tests for the endpoints.

5.2 Taylor Series

The Taylor series of $f(x)$ around $x = a$ is the power series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} (x-a)^2 + \frac{f'''(a)}{3!} (x-a)^3 + \cdots .$$

When $a = 0$, it is also called the Maclaurin series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots .$$

5.3 Some Common Taylor Series

$$\begin{aligned}e^x &= \sum_{k=0}^{\infty} \frac{x^k}{k!} && \text{(converges for all } x\text{)} \\ \sin(x) &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} && \text{(converges for all } x\text{)} \\ \cos(x) &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} && \text{(converges for all } x\text{)} \\ \ln(1-x) &= -\sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} && \text{(converges for } |x| < 1\text{)}\end{aligned}$$

5.4 Manipulating Taylor Series

If you know a Taylor series

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots,$$

you can compute the series for $f'(x)$ by differentiating each term, and you can compute the series for $\int f(x) dx$ by integrating each term.

5.5 Taylor Polynomials and Error Estimates

The n 'th Taylor polynomial is

$$\begin{aligned}T_n(x) &= \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n.\end{aligned}$$

The error

$$R_n(x) = f(x) - T_n(x)$$

in using $T_n(x)$ to estimate $f(x)$ is bounded by

$$|R_n(x)| \leq M \frac{|x-a|^{n+1}}{(n+1)!},$$

where M is any number so that $|f^{(n+1)}(t)| \leq M$ for all t between x and a .

5.6 Euler's Identity

$$e^{i\theta} = \cos \theta + i \sin \theta, \quad \text{where } i = \sqrt{-1}.$$

6 Differential Equations

A *differential equation* is an equation that involves x , y , and one or more derivatives of y . The *order* of a differential equation is the highest derivative that appears.

A function $y = f(x)$ is a solution to a differential equation if substituting $y = f(x)$ makes the equation true.

6.1 Slope Fields

A differential equation of the form

$$\frac{dy}{dx} = f(x, y)$$

can be studied by drawing the slope field, which means at each point (x_0, y_0) , drawing a small line segment through the point having slope $f(x_0, y_0)$.

6.2 First Order Linear Differential Equations

A first order linear differential equation has the form

$$\frac{dy}{dx} + P(x)y = Q(x).$$

To solve, compute the integrating factor

$$v(x) = e^{\int P(x) dx}.$$

Then multiplying by $v(x)$ makes the left-hand side of the equation equal to $\frac{d}{dx}(v(x)y)$, so the solution is

$$y = \frac{1}{v(x)} \int v(x)Q(x) dx.$$

6.3 Applications of First Order Linear Differential Equations

Population growth and radioactive decay problems lead to separable differential equations of the form

$$\frac{dy}{dx} = ky \quad \text{with solution} \quad y = Ce^{kx}.$$

Mixing problems in which liquid containing a chemical is pouring into and drained out of a container often lead to general first order linear differential equations.

6.4 Euler's Method

To approximately solve

$$\frac{dy}{dx} = f(x, y),$$

start at a point (x_0, y_0) , choose a (small) increment value, and then for $n = 1, 2, 3, \dots$ let

$$x_n = x_{n-1} + dx \quad \text{and} \quad y_n = y_{n-1} + f(x_{n-1}, y_{n-1}) dx.$$

An improved method is to let

$$\begin{aligned} x_n &= x_{n-1} + dx, \\ z_n &= y_{n-1} + f(x_{n-1}, y_{n-1}) dx, \\ y_n &= y_{n-1} + \left(\frac{f(x_{n-1}, y_{n-1}) + f(x_n, z_n)}{2} \right) dx. \end{aligned}$$

6.5 Second Order Linear Differential Equations

A second order linear differential equation has the form

$$P(x) \frac{d^2y}{dx^2} + Q(x) \frac{dy}{dx} + R(x)y = G(x).$$

If $G(x) = 0$, the equation is homogeneous.

6.6 Homogeneous Second Order Linear Differential Equations with Constant Coefficients

To solve

$$ay'' + by' + cy = 0$$

when a and b are constants, first find the roots of

$$ar^2 + br + c = 0. \quad \text{The roots are } r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2}.$$

There are three cases.

Case I: $b^2 - 4ac > 0$: Then r_1 and r_2 are real, and the general solution is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

Case II: $b^2 - 4ac < 0$: Then r_1 and r_2 are complex numbers, so we can write them as

$$r_1 = \alpha + \beta i \quad \text{and} \quad r_2 = \alpha - \beta i.$$

Then the general solution is

$$y = e^{\alpha x} (c_1 \cos(\beta x) + c_2 \sin(\beta x)).$$

Case III: $b^2 - 4ac = 0$: Then $r_1 = r_2 = -\frac{b}{2}$, and the general solution is

$$y = c_1 e^{r_1 x} + c_2 x e^{r_1 x} = (c_1 + c_2 x) e^{r_1 x}.$$

6.7 Non-Homogeneous Second Order Linear Differential Equations with Constant Coefficients

To find the general solution to

$$ay'' + by' + cy = G(x),$$

first find the solution $y_c(x)$ to the complementary homogeneous equation

$$ay'' + by' + cy = 0.$$

Next find a particular solution $y_p(x)$ to the non-homogeneous equation. Then the general solution to the non-homogeneous equation is $y(x) = y_c(x) + y_p(x)$.

6.8 Method of Undetermined Coefficients

If $G(x)$ is a polynomial, or e^{rx} or a trig function multiplied by a polynomial, one can guess the form of the solution, leaving the coefficients as unknowns, substitute into the differential equation, and solve for the coefficients.

6.9 Method of Variation of Parameters

Let $y_c(x) = c_1y_1(x) + c_2y_2(x)$ be the solution to the complementary homogeneous equation. Solve the equations

$$v_1'(x)y_1(x) + v_2'(x)y_2(x) = 0 \quad \text{and} \quad v_1'(x)y_1'(x) + v_2'(x)y_2'(x) = G(x)/a$$

for $v_1'(x)$ and $v_2'(x)$. Integrate to find $v_1(x)$ and $v_2(x)$. Then $y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$.

6.10 Harmonic Motion

If a weight of mass m is attached to a spring having spring constant k , and if it moves through a medium with friction equal to δ times its instantaneous velocity, then the equation of motion is

$$my'' + \delta y' + ky = 0.$$

Depending on the values of m , δ , and k , the solution will involve exponentials and/or trig functions.

6.11 Series Solutions to Differential Equations

To find a series solution to

$$P(x)y'' + Q(x)y' + R(x)y = 0,$$

substitute

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

into the equation, set all power of x equal to zero, and find relations on c_0, c_1, c_2, \dots . In general, there should be two coefficients of $y(x)$ that can be chosen arbitrarily, and then all of the other coefficients can be expressed in terms of those two.

7 Polar Coordinates

Polar coordinates (r, θ) describe a point whose distance from the origin is r and whose line segment makes an angle θ with the positive x -axis. Polar and Cartesian coordinates are related by the formulas:

$$x = r \cos \theta, \quad y = r \sin \theta, \quad x^2 + y^2 = r^2, \quad \frac{y}{x} = \tan \theta.$$

7.1 Graphing in Polar Coordinates

Methods to use include:

- Check for symmetries. What happens when you replace r by $-r$ and/or replace θ by $-\theta$?
- Make a table of values.
- Convert to Cartesian coordinates.

7.2 Areas in Polar Coordinates

The area inside the curve $r = f(\theta)$ with $\alpha \leq \theta \leq \beta$ is

$$A = \int_{\alpha}^{\beta} \frac{1}{2} f(\theta)^2 d\theta.$$

8 Parametric Equations

We can describe a curve using parametric equations

$$x = f(t) \quad \text{and} \quad y = g(t).$$

The slope at $t = t_0$ is

$$\left. \frac{dy}{dx} \right|_{t=t_0} = \frac{(dy/dt)|_{t=t_0}}{(dx/dt)|_{t=t_0}}.$$