Problem 1. (10 points) (a) Let A and B be the matrices

$$A = \begin{pmatrix} 3 & x \\ -2 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 6 & 6 \\ -8 & -2 \end{pmatrix}$$

Find all values of x for which $AB = BA$. $x =$ $\boxed{}$

(b) Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be the vectors

$$\boldsymbol{a} = (1, -1, 3) \quad \text{and} \quad \boldsymbol{b} = (2, 1, -1).$$

Compute the cross product $\boldsymbol{a} \times \boldsymbol{b}$. $\boldsymbol{a} \times \boldsymbol{b} =$ $\boxed{}$

Solution. (a) Compute

$$AB = \begin{pmatrix} -8x + 18 & -2x + 18 \\ -20 & -14 \end{pmatrix} \quad \text{and} \quad BA = \begin{pmatrix} 6 & 6x + 6 \\ -20 & -8x - 2 \end{pmatrix}$$

Setting $AB = BA$ gives four equations

$$\begin{aligned} -8x + 18 &= 6 \\ -2x + 18 &= 6x + 6 \\ -20 &= -20 \\ -14 &= -8x - 2. \end{aligned}$$

Solving the first equation for x gives $\boxed{x = \frac{3}{2}}$, and then one checks that $x = \frac{3}{2}$ is also a solution to the other three equations.

(b)

$$\begin{aligned} \boldsymbol{a} \times \boldsymbol{b} &= \det \begin{pmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & -1 & 3 \\ 2 & 1 & -1 \end{pmatrix} \\ &= \det \begin{pmatrix} -1 & 3 \\ 1 & -1 \end{pmatrix} \boldsymbol{i} - \det \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} \boldsymbol{j} + \det \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \boldsymbol{k} \\ &= \boxed{-2\boldsymbol{i} + 7\boldsymbol{j} + 3\boldsymbol{k}}. \end{aligned}$$

Problem 2. (10 points) (a) Let $F(x, y, z) = x^5 e^{y^2} \cos(z)$. Compute

$$\frac{\partial^5 F}{\partial x^2 \partial y \partial z^2} = \boxed{}$$

(b) We consider two functions $f(u,v)$ and $g(x,y)$. The function g is given by the formula

$$g(x,y) = (x^2 + y, x - y^2).$$

The values of $f(u,v)$ and its partial derivatives at various points (u,v) are given in the following table:

(u,v)	$(1,1)$	$(2,1)$	$(3,1)$	$(4,1)$	$(5,1)$	$(6,1)$
$f(u,v)$	A	B	C	D	E	F
$\frac{\partial f}{\partial u}(u,v)$	G	H	I	J	K	L
$\frac{\partial f}{\partial v}(u,v)$	M	N	O	P	Q	R

Compute the value of

$$\frac{\partial(f \circ g)}{\partial x}(2,1).$$

Express your answer in terms of the quantities $A, B, C, \ldots, Q, R$. $\frac{\partial(f \circ g)}{\partial x}(2,1) =$

Solution. (a) We just need to differentiate twice for the x variable, once for the y variable, and twice for the z variable, so

$$\begin{aligned}\frac{\partial^5 F}{\partial x^2 \partial y \partial z^2} &= \frac{\partial^5}{\partial x^2 \partial y \partial z^2}\big(x^5 e^{y^2}\cos(z)\big)\\ &= 20x^3 \cdot 2ye^{y^2} \cdot (-\cos(z))\\ &= \boxed{-40x^3 y e^{y^2}\cos(z)}\end{aligned}$$

(b) We write

$$g(x,y) = \big(u(x,y), v(x,y)\big) = (x^2 + y, x - y^2).$$

Note that

$$g(2,1) = (4+1, 2-1) = (5,1).$$

We use the chain rule to compute

$$\begin{aligned}\frac{\partial(f \circ g)}{\partial x}(2,1) &= \frac{\partial f}{\partial u}\big(g(2,1)\big) \cdot \frac{\partial u}{\partial x}(2,1) + \frac{\partial f}{\partial v}\big(g(2,1)\big) \cdot \frac{\partial v}{\partial x}(2,1)\\ &= \frac{\partial f}{\partial u}(5,1) \cdot 4 + \frac{\partial f}{\partial v}(5,1) \cdot 1\\ &= \boxed{4K + Q}\end{aligned}$$

Problem 3. (10 points) Suppose that we know that the level surface

$$ax^2 + by^3 + z^4 = k$$

goes through the point $(1, 1, 1)$, and we also know that the tangent plane at $(1, 1, 1)$ is orthogonal to the vector $-2\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$. What are the values of a and b and k? (Hint: First find a and b, then k.)

$a =$ [ ] and $b =$ [ ] and $k =$ [ ]

Solution. Let $f(x, y, z) = ax^2 + by^3 + z^4$. The gradient vector

$$\begin{aligned}\nabla f(1,1,1) &= f_x(1,1,1)\boldsymbol{i} + f_y(1,1,1)\boldsymbol{j} + f_z(1,1,1)\boldsymbol{k} \\ &= 2a\boldsymbol{i} + 3b\boldsymbol{j} + 4\boldsymbol{k}\end{aligned}$$

is orthogonal to the tangent plane to the level surface at the point $(1, 1, 1)$, so the gradient vector and $-2\boldsymbol{i}+2\boldsymbol{j}+2\boldsymbol{k}$ are parallel. That means that $2a\boldsymbol{i} + 3b\boldsymbol{j} + 4\boldsymbol{k}$ and $-2\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$ are scalar multiples of one another, so there is some number t so that

$$2a\boldsymbol{i} + 3b\boldsymbol{j} + 4\boldsymbol{k} = t(-2\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}) = -2t\boldsymbol{i} + 2t\boldsymbol{j} + 2t\boldsymbol{k}.$$

Looking at the $\boldsymbol{k}$ coordinate, we see that $4 = 2t$, so $t = 2$. Then the first and second coordinates say that $2a = -4$ and $3b = 4$, and hence $\boxed{a = -2 \text{ and } b = \frac{4}{3}}$. Finally, we know that the point $(1, 1, 1)$ is on the surface $ax^2 + by^3 + z^4 = k$, and we know the values of a and b, so

$$k = a \cdot 1^2 + b \cdot 1^3 + 1^4 = a + b + 1 = -2 + \frac{4}{3} + 1 = \boxed{\frac{1}{3}}$$

Problem 4. (10 points)
NOTE: Grading for each part of this True/False question is +2 for the correct answer, 0 if left blank, and −1 if incorrect.

The values of a (twice differentiable) function f and its partial derivatives at various points are given in the following table.

	P	Q	R
f	3	−7	2
f_x	0	0	0
f_y	0	1	0
f_{xx}	−2	3	−2
f_{xy}	3	2	3
f_{yy}	−5	4	5

Indicate whether each of the following statements is true or false by circling the appropriate answer. You do not need to give a reason for your answer.

Solution.

(a) P is a critical point **True**

(b) Q is a local minimum **False**

(c) Q is a saddle point **False**

(d) R is a saddle point **True**

(e) P is a local maximum **True**

I didn't ask you to give reasons for your answers, but here are the reasons.

(a) P is a critical point because $f_x(P) = 0$ and $f_y(P) = 0$.

(b) Q is not a local minimum, since it is not even a critical point, since $f_y(Q) = 1 \neq 0$.

(c) Q is also not a saddle point, since it is not a critical point, since $f_y(Q) = 1 \neq 0$.

(d) R is a saddle point since

$$f_{xx}(R)f_{yy}(R) - f_{xy}(R)^2 = (-2) \cdot 5 - 3^2 = -19 < 0.$$

(e) P is a local maximum since

$$f_{xx}(P)f_{yy}(P) - f_{xy}(P)^2 = (-2) \cdot (-5) - 3^2 = 1 > 0$$

and $f_{xx}(P) = -2 < 0$. (These are the conditions for the 2nd derivative test.)

Problem 5. (10 points) Write down the second order Taylor expansion of the function

$$f(x, y) = xe^{3xy} \quad \text{around the point } (1, 0).$$

To help, I've computed some of the partial derivatives for you, and you can fill in the others:

$$\begin{aligned} f_x(x, y) &= e^{3xy} + 3xye^{3xy} \\ f_y(x, y) &= \underline{\hspace{3cm}} \\ f_{xx}(x, y) &= 6ye^{3xy} + 9xy^2e^{3xy} \\ f_{xy}(x, y) &= 6xe^{3xy} + 3x^2ye^{3xy} \\ f_{yy}(x, y) &= \underline{\hspace{3cm}} \end{aligned}$$

Solution. First we compute the partial derivatives:

$$\begin{aligned}
f(x,y) &= xe^{3xy}\\
f_x(x,y) &= e^{3xy} + 3xye^{3xy}\\
f_y(x,y) &= 3x^2e^{3xy}\\
f_{xx}(x,y) &= 3ye^{3xy} + 3ye^{3xy} + 9xy^2e^{3xy} = 6ye^{3xy} + 9xy^2e^{3xy}\\
f_{xy}(x,y) &= f_{xy}(x,y) = 6xe^{3xy} + 3x^2ye^{3xy}\\
f_{yy}(x,y) &= 9x^3e^{3xy}
\end{aligned}$$

Then we evaluate the derivatives at the point $(1,0)$.

$$\begin{aligned}
f(x,y) &= 1\\
f_x(x,y) &= 1\\
f_y(x,y) &= 3\\
f_{xx}(x,y) &= 0\\
f_{xy}(x,y) &= f_{xy}(x,y) = 6\\
f_{yy}(x,y) &= 9
\end{aligned}$$

Finally, we use these values in Taylor's formula:

$$f(1+h_1,h_2) = 1 + h_1 + 3h_2 + \frac{1}{2}(0h_1^2 + 6h_1h_2 + 6h_2h_1 + 9h_2^2).$$

Simplifying this last expression gives

$$\boxed{f(1+h_1,h_2) = 1 + h_1 + 3h_2 + 6h_1h_2 + \tfrac{9}{2}h_2^2}$$