
Problem 1. (10 points) Write the following iterated integral as a sum of one or more iterated integrals with the order of integration reversed, i.e., each inner integral should be with respect to x and each outer integral should be with respect to y.

$$\int_0^\pi \int_{\sin x}^2 f(x,y)\,dy\,dx.$$

Also sketch the region.

Solution. The region lies over the curve $y = \sin x$ and under the curve $y = 2$. In order to integrate in the other order, we split it into three regions. (Note that the values of $\sin^{-1}(y)$ are between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$.)

$$\begin{aligned} D_1 &= \{(x,y) : 1 \le y \le 2,\ 0 \le x \le \pi\} \\ D_2 &= \{(x,y) : 0 \le y \le 1,\ 0 \le x \le \sin^{-1}(y)\} \\ D_3 &= \{(x,y) : 0 \le y \le 1,\ \pi - \sin^{-1}(y) \le x \le \pi\} \end{aligned}$$

Hence

$$\begin{aligned} \int_0^\pi \int_{\sin x}^2 f(x,y)\,dy\,dx &= \int_1^2 \int_0^\pi f(x,y)\,dx\,dy \\ &\quad + \int_0^1 \int_0^{\sin^{-1}(y)} f(x,y)\,dx\,dy + \int_0^1 \int_{\pi-\sin^{-1}(y)}^{\pi} f(x,y)\,dx\,dy \end{aligned}$$

Problem 2. (10 points) Evaluate the following integral.

$$\iint_D \cos(x^2+y^2)\,dA, \quad \text{where } D = \{(x,y) : 1 \le x^2+y^2 \le 9\}.$$

Solution. We use polar coordinates, so

$$x = r\cos\theta, \quad y = r\sin\theta, \quad dA = dx\,dy = r\,dr\,d\theta, \quad x^2+y^2 = r^2.$$

Also $D^* = \{(r,\theta) : 1 \le r \le 3 \text{ and } 0 \le \theta \le 2\pi\}$. So

$$\begin{aligned} \iint_D \cos(x^2+y^2)\,dx\,dy &= \iint_{D^*} \cos(r^2)\,r\,dr\,d\theta \\ &= \int_0^{2\pi} \int_1^3 \cos(r^2)\,r\,dr\,d\theta \\ &= \int_0^{2\pi} \frac{1}{2}\sin(r^2)\Big|_{r=1}^{r=3}\,d\theta \end{aligned}$$

$$= \int_0^{2\pi} \frac{1}{2}\big(\sin(9) - \sin(1)\big)\, d\theta$$
$$= \pi\big(\sin(9) - \sin(1)\big).$$

Problem 3. (10 points) The curve

$$\boldsymbol{c}(t) = (t^{-3}, t^{-2}, t^{-1})$$

for $t > 0$ is a flow line for the vector field

$$\boldsymbol{F}(x, y, z) = (axz + y^2, byz - x, cy + z^2).$$

What are the values of a, b, and c?

Solution. We set $\boldsymbol{F}(\boldsymbol{c}(t)) = \boldsymbol{c}'(t)$. We have

$$\begin{aligned} \boldsymbol{F}(\boldsymbol{c}(t)) &= \boldsymbol{F}(t^{-3}, t^{-2}, t^{-1}) \\ &= (at^{-4} + t^{-4}, bt^{-3} - t^{-3}, ct^{-2} + t^{-2}) \\ &= \big((a+1)t^{-4}, (b-1)t^{-3}, (c+1)t^{-2})\big), \\ \boldsymbol{c}'(t) &= (-3t^{-4}, -2t^{-3}, -t^{-2}). \end{aligned}$$

These need to be equal for all $t > 0$, so we get

$$a + 1 = -3, \quad b - 1 = -2, \quad c + 1 = -1,$$

which gives

$$\boxed{a = -4, \quad b = -1, \quad c = -2}$$

Problem 4. (10 points)
NOTE: Grading for each part of this True/False question is +2 for the correct answer, 0 if left blank, and −1 if incorrect.

Indicate whether each of the following statements is true or false by circling the appropriate answer. You do **not** need to give a reason for your answer. For (a), (b), and (c), we write $\boldsymbol{F}(x, y, z)$ for a 3-dimensional vector field that is assumed to be sufficiently differentiable.

Solution.

(a)	$\mathrm{curl}(\mathrm{curl}(\boldsymbol{F})) = \mathbf{0}$ for all $\boldsymbol{F}$	**FALSE**
(b)	$\mathrm{curl}(\mathrm{div}(\boldsymbol{F})) = \mathbf{0}$ for all $\boldsymbol{F}$	**FALSE**
(c)	$\mathrm{div}(\mathrm{curl}(\boldsymbol{F})) = 0$ for all $\boldsymbol{F}$	**TRUE**
(d)	$\boldsymbol{F} = (y, x, z)$ is a gradient vector field	**TRUE**
(e)	$\boldsymbol{F} = (z, x, y)$ is a gradient vector field	**FALSE**

Here are the reasons.
(a) For cross products of vectors, $\boldsymbol{a} \times (\boldsymbol{a} \times \boldsymbol{b})$ is a vector in the plane spanned by $\boldsymbol{a}$ and $\boldsymbol{b}$ and perpendicular to $\boldsymbol{a}$, but it won't in general be zero. Similarly for the curl of a curl. It's easy enough to find an example that gives a non-zero result. For example,

$$\boldsymbol{F}(x, y, z) = (y^2, 0, 0)$$

has

$$\operatorname{curl} \boldsymbol{F} = (0, 0, -2y),$$

so

$$\operatorname{curl}(\operatorname{curl}(\boldsymbol{F})) = (-2, 0, 0)$$

is non-zero.
(b) The quantity $\operatorname{curl}(\operatorname{div}(\boldsymbol{F}))$ isn't even defined, since the divergence $\operatorname{div}(\boldsymbol{F})$ is a real-valued function, not a vector field, so you can't take its curl.
(c) This one is true, as is the fact that the curl of a gradient is zero.
(d) This passes the partial derivative tests. And it's easy enough to find that $\boldsymbol{F}$ is the gradient of the function $f(x, y, z) = xy + \frac{1}{2}z^2$.
(e) This is not a gradient, since for example,

$$\frac{\partial F_1}{\partial z} = \frac{\partial z}{\partial z} = 1$$

is not equal to

$$\frac{\partial F_3}{\partial x} = \frac{\partial y}{\partial x} = 0.$$

Problem 5. (10 points) Let D^* be the square $[0, 1] \times [0, 1]$ in the uv-plane, and let T be the function

$$T(u, v) = (u^2 - v^2, 2uv).$$

(a) Sketch the region $T(D^*)$ in the xy-plane. (*Hint.* Figure out where the sides of the square are sent.)
(b) If

$$\int\int_{T(D^*)} \frac{1}{\sqrt{x^2 + y^2}}\, dx\, dy \int\int_{D^*} f(u, v)\, du\, dv,$$

what is the function $f(u, v)$? (You may assume that $T : D^* \to \mathbb{R}^2$ is one-to-one.)
(c) Evaluate the integral in (b).

Solution. (b) The Jacobian is

$$\frac{\partial(x, y)}{\partial(u, v)} = \det \begin{pmatrix} 2u & -2v \\ 2v & 2u \end{pmatrix} = 4u^2 + 4v^2,$$

so

$$\int\int_{T(D^*)} \frac{1}{\sqrt{x^2+y^2}}\,dx\,dy$$
$$= \int\int_{D^*} \frac{1}{\sqrt{(u^2-v^2)^2+(2uv)^2}}\,(4u^2+4v^2)\,du\,dv.$$

This gives an answer to (b) with

$$f(u,v) = \frac{4u^2+4v^2}{\sqrt{(u^2-v^2)^2+(2uv)^2}}.$$

However, for answering (c), we should simplify by noticing that

$$\begin{aligned}(u^2-v^2)^2+(2uv)^2 &= u^4-2u^2v^2+v^4+4u^2v^2\\ &= u^4+2u^2v^2+v^4\\ &= (u^2+v^2)^2.\end{aligned}$$

So

$$f(u,v) = \frac{4u^2+4v^2}{\sqrt{(u^2+v^2)^2}} = 4.$$

(c)

$$\int\int_{T(D^*)} \frac{1}{\sqrt{x^2+y^2}}\,dx\,dy = \int\int_{D^*} 4\,du\,dv = 4\text{Area}(D^*) = \boxed{4}.$$