Problem 1. (10 points) (a) Compute the line integral

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{s}$$

for the path $\boldsymbol{c}(t) = (t^2, t^3, t)$ with $0 \le t \le 1$ and the vector field $\boldsymbol{F}(x, y, z) = x\boldsymbol{i} + z\boldsymbol{j} + x\boldsymbol{k}$.

(b) Compute the line integral

$$\int_C z\,dx + y\,dy + x\,dz$$

for the path $\boldsymbol{c}(t) = \big(e^{t^2}, \ln(t+1), \cos(t)\big)$ with $0 \le t \le 1$.

Solution. (a) We have $\boldsymbol{c}'(t) = (2t, 3t^2, 1)$, so

$$\begin{aligned}\int_C \boldsymbol{F} \cdot d\boldsymbol{s} &= \int_0^1 \boldsymbol{F}(t^2, t^3, t) \cdot \boldsymbol{c}'(t)\,dt \\ &= \int_0^1 (t^2, t, t^2) \cdot (2t, 3t^2, 1)\,dt \\ &= \int_0^1 2t^3 + 3t^3 + t^2\,dt \\ &= \frac{5}{4}t^4 + \frac{1}{3}t^3 \bigg|_{t=0}^{t=1} \\ &= \frac{5}{4} + \frac{1}{3} = \boxed{\frac{19}{12}}\end{aligned}$$

(b) This is the integral of the vector field

$$\boldsymbol{F}(x, y, z) = z\boldsymbol{i} + y\boldsymbol{j} + x\boldsymbol{k}.$$

This vector field satisfies the conditions to be a gradient field, and it's easy enough to find that

$$\boldsymbol{F} = \nabla f \quad \text{for the function} \quad f(x, y, z) = xz + \frac{1}{2}y^2.$$

The fundamental theorem of calculus for line integrals says that the value of the integral is given by the difference of the values of f at the endpoints of the curve. So

$$\begin{aligned}\int_C z\,dx + y\,dy + x\,dz &= \int_C \boldsymbol{F} \cdot d\boldsymbol{s} \\ &= \int_C (\nabla f) \cdot d\boldsymbol{s} \\ &= f\big(\boldsymbol{c}(1)\big) - f\big(\boldsymbol{c}(0)\big) \\ &= f\big(e, \ln(2), \cos(1)\big) - f(1, 0, 1)\end{aligned}$$

$$= \boxed{e\cos(1) + \frac{1}{2}(\ln 2)^2 - 1}$$

Problem 2. (15 points) Let D be the region

$$D = \Big\{(x,y) : 0 \le x \le 2 \text{ and } y \ge 0 \text{ and } 1 \le x^2 + y^2 \le 9\Big\}.$$

(a) Sketch the region D.
(b) Write the integral

$$\int_D f(x,y)\,dx\,dy$$

as a sum of one or more iterated integrals in xy-coordinates.
(c) Write the integral

$$\int_D f(x,y)\,dx\,dy$$

as a sum of one or more iterated integrals in polar coordinates.

Solution. (b) For $0 \le x \le 1$, the region is $\sqrt{1-x^2} \le y \le \sqrt{9-x^2}$, while for $1 \le x \le 2$, the region is $0 \le y \le \sqrt{9-x^2}$. So

$$\int_D f(x,y)\,dx\,dy = \int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{9-x^2}} f(x,y)\,dx\,dy + \int_1^2 \int_0^{\sqrt{9-x^2}} f(x,y)\,dx\,dy$$

(c) The vertical line $x = 2$ intersects the circle $x^2+y^2 = 9$ at the point whose angle θ is $\cos^{-1}(2/3)$. So for $0 \le \theta \le \cos^{-1}(-2/3)$, the values of r go from $r = 1$ to the line $x = 2$. Since $x = r\cos\theta$, that means that r goes from 1 to $2/\cos\theta$. Then, for $\cos^{-1}(2/3) \le \theta \le \pi/2$, the value of r goes from 1 to 3. Hence

$$\int_D f(x,y)\,dx\,dy = \int_0^{\cos^{-1}(2/3)} \int_1^{2/\cos\theta} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta + \int_{\cos^{-1}(2/3)}^{\pi/2} \int_1^3 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

Problem 3. (10 points) Find all of the critical points of the function

$$f(x,y) = \frac{1}{3}x^3 + \frac{1}{3}y^3 - \frac{1}{2}x^2 - \frac{5}{2}y^2 + 6y + 10$$

and classify the critical points as local maxima, local minima, and saddle points.

Solution. We have

$$f_x(x,y) = x^2 - x = x(x-1),$$
$$f_y(x,y) = y^2 - 5y + 6 = (y-2)(y-3).$$

So there are four critical points:

$$(0,2),\ (0,3),\ (1,2),\ (1,3).$$

For each one we need to compute

$$D = f_{xx}f_{yy} - f_{xy}^2 = (2x-1)(2y-5) - 0^2 = (2x-1)(2y-5).$$

Then a point is a local minimum if $D > 0$ and $f_{xx} > 0$, it is a local maximum if $D > 0$ and $f_{xx} < 0$, and it is a saddle point if $D < 0$. Note that $f_{xx} = 2x - 1$. We make a little table:

Point	$(0,2)$	$(0,3)$	$(1,2)$	$(1,3)$
Value of D	1	-1	-1	1
Value of f_{xx}	-1			1
Type of point	Max	Saddle	Saddle	Min

Problem 4. (10 points) Let $f(x,y)$ be defined by

$$f(x,y) = \begin{cases} \dfrac{2x^3 - 3y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Calculate $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$ directly from the definition.

(b) Let a and b be non-zero constants, and define a function

$$g(t) = f(at, bt). \quad \text{Calculate } \frac{dg}{dt}(0).$$

(c) Let $\boldsymbol{h}(t) = (at, bt)$, so the function $g(t)$ in (b) is $g(t) = f(\boldsymbol{h}(t))$. The chain rule would say that

$$\frac{dg}{dt}(0) = \nabla f(0,0) \cdot \boldsymbol{h}'(0) = \frac{\partial f}{\partial x}(0,0)a + \frac{\partial f}{\partial y}(0,0)b.$$

Does this agree with your answers from parts (a) and (b)? If not, explain what is going wrong.

Solution. (a) We compute

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} \qquad \text{definition of partial derivative,}$$
$$= \lim_{h\to 0} \frac{2h^3/h^2}{h} \qquad \text{definition of } f,$$

$$= \boxed{2}.$$

Similarly,

$$\begin{aligned}
\frac{\partial f}{\partial y}(0,0) &= \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{h} && \text{definition of partial derivative,}\\
&= \lim_{k\to 0} \frac{-3k^3/k^2}{k} && \text{definition of } f,\\
&= \boxed{-3}.
\end{aligned}$$

(b) For $t \neq 0$ we have

$$g(t) = f(at, bt) = \frac{2(at)^3 - 3(bt)^3}{(at)^2 + (bt)^2} = \frac{2a^3 - 3b^3}{a^2 + b^2} t.$$

This formula is also true for $t = 0$, since $g(0) = f(0,0) = 0$. Hence

$$\boxed{g'(0) = \frac{2a^3 - 3b^3}{a^2 + b^2}}$$

(In fact, this is $g'(t)$ for every value of t.)

(c) From (b) we have

$$g'(0) = \frac{2a^3 - 3b^3}{a^2 + b^2}.$$

But using (a) we have

$$\frac{\partial f}{\partial x}(0,0) \cdot a + \frac{\partial f}{\partial y}(0,0) \cdot b = 2a - 3b.$$

These are not the same in general. Indeed, their difference is

$$\frac{2a^3 - 3b^3}{a^2 + b^2} - (2a - 3b) = \frac{-2ab^2 + 3a^2 b}{a^2 + b^2} = \frac{ab(-2b + 3a)}{a^2 + b^2},$$

so they are the same only if $a = 0$, $b = 0$, or $3a = 2b$. The reason that this does not contradict the chain rule is because the chain rule only applies if the partial derivatives are continuous. In this example, the partial derivatives of f, although they do exist at $(0,0)$, are not continuous.

Problem 5. (15 points) For each of the following vector fields $\boldsymbol{F}$, check whether $\boldsymbol{F}$ is conservative.[1] If it is conservative, find a potential function. If it is not conservative, explain why not.

(a) $\boldsymbol{F} = z\boldsymbol{i} + \left(x^2 + \frac{1}{2}z^2\right)\boldsymbol{j} + (x + yz)\boldsymbol{k}$.

(b) $\boldsymbol{F} = (2xy + \frac{1}{2}x)\boldsymbol{i} + (x^2 + \sin^2 3y)\boldsymbol{j}$.

[1] Note: Despite the new majorities in the House and the Senate, there is not yet a law saying that all (American) vector fields are conservative!

(c) Let $\boldsymbol{a}$ be a non-zero constant vector, let $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, and let $\boldsymbol{F} = \boldsymbol{a} \times \boldsymbol{r}$.

Solution. By definition, a vector field $\boldsymbol{F}$ is conservative if it is the gradient of a function $\boldsymbol{F} = \nabla f$.
(a) A vector field $\boldsymbol{F} = P\boldsymbol{i} + Q\boldsymbol{j} + R\boldsymbol{k}$ defined everywhere on a solid region (or even defined everywhere except for a finite set of points) is conservative if and only if its curl is zero, or equivalently, if

$$P_y = Q_x \quad \text{and} \quad P_z = R_x \quad \text{and} \quad Q_z = R_y.$$

In this case we have

$$\begin{aligned} P_y &= 0 \quad \text{and} \quad Q_x = 2x, \\ P_z &= 1 \quad \text{and} \quad R_x = 1, \\ Q_z &= z \quad \text{and} \quad R_y = z. \end{aligned}$$

The first line shows that $\boxed{\boldsymbol{F} \text{ is not conservative}}$. Alternatively, one computes $\operatorname{curl}(\boldsymbol{F}) = 2x\boldsymbol{i}$ is nonzero.
(b) Similarly, a vector field $\boldsymbol{F} = P\boldsymbol{i} + Q\boldsymbol{j}$ in the plane that is defined everywhere in a region is conservative if and only if $Q_x = P_y$. In this case

$$Q_x(x,y) = 2x = P_y(x,y),$$

so $\boxed{\boldsymbol{F} \text{ is conservative}}$. We can find an $f(x,y)$ by inspection, or more systematically by integration. Thus if $\boldsymbol{F} = \nabla f$, then

$$f_x(x,y) = P(x,y) = 2xy + \frac{1}{2}x.$$

Integrating with respect to x gives

$$f(x,y) = x^2 y + \frac{1}{4}x^2 + g(y)$$

for some function $g(y)$ depending only on y. Then we use

$$x^2 + g'(y) = f_y(x,y) = Q(x,y) = x^2 + \sin^2(3y)$$

to find that $g'(y) = \sin^2(3y)$. So now we just need to integrate

$$g(y) = \int \sin^2(3y)\,dy = \int \frac{1 - \cos(6y)}{2}\,dy = \frac{y}{2} - \frac{\sin(6y)}{12}.$$

Using this in our formula for $f(x,y)$ gives the desired function,

$$\boxed{f(x,y) = x^2 y + \frac{1}{4}x^2 + \frac{y}{2} - \frac{\sin(6y)}{12}}$$

Of course, one can always add a constant.

(c) Let $\boldsymbol{a} = (a, b, c)$. Then

$$\boldsymbol{F} = \boldsymbol{a} \times \boldsymbol{r} = \det \begin{pmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a & b & c \\ x & y & z \end{pmatrix} = (bz - cy)\boldsymbol{i} - (az - cx)\boldsymbol{j} + (ay - bx)\boldsymbol{k}.$$

As in (a), we need to check if the curl vanishes. For this vector field, we have

$$\nabla \times \boldsymbol{F} = \det \begin{pmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ bz - cy & -az + cx & ay - bx \end{pmatrix} = 2a\boldsymbol{i} + 2b\boldsymbol{j} + 2c\boldsymbol{k}.$$

So the curl of this vector field $\boldsymbol{F}$ is constant, and indeed is given by $\nabla \times \boldsymbol{F} = 2\boldsymbol{a}$. Since this is non-zero, $\boxed{\boldsymbol{F} \text{ is not conservative}}$.

Problem 6. (10 points) Let C be the unit circle

$$C = \{(x, y) : x^2 + y^2 = 1\}$$

oriented in a counter-clockwise direction. Let $f(t)$ and $g(t)$ be functions of one variable with continuous derivatives. Evaluate

$$\int_C \big(f(x) + g(y)\big)\,dx + \big(xg'(y) + 3x - 7\big)\,dy.$$

Solution. The easiest way to do this problem is to let D be the unit disk, so $C = \partial D$, and use Green's theorem. Thus

$$\begin{aligned}
&\int_C \big(f(x) + g(y)\big)\,dx + \big(xg'(y) + 3x - 7\big)\,dy \\
&\quad = \int_{\partial D} \big(f(x) + g(y)\big)\,dx + \big(xg'(y) + 3x - 7\big)\,dy \\
&\quad = \iint_D \frac{\partial}{\partial x}\big(xg'(y) + 3x - 7\big) - \frac{\partial}{\partial y}\big(f(x) + g(y)\big)\,dx\,dy \\
&\qquad\qquad\qquad \text{using Green's theorem,} \\
&\quad = \iint_D \big(g'(y) + 3\big) - g'(y)\,dx\,dy \\
&\quad = \iint_D 3\,dx\,dy \\
&\quad = 3\text{Area}(D) \\
&\quad = \boxed{3\pi}
\end{aligned}$$

Problem 7. (10 points) Let S be a surface in $\mathbb{R}^3$, and let ∂S be the boundary of S. Let $\boldsymbol{F}$ be a vector field on S with continuous partial derivatives. Suppose that you are given the following information about S and $\boldsymbol{F}$:

(i) S lies in the plane $y = 3$

(ii) $\text{Area}(S) = 17$

(iii) $\text{Length}(\partial S) = 25$

(iv) $\text{div}(\boldsymbol{F}) = x^2 + y^2 - z$

(v) $\text{curl}(\boldsymbol{F}) = 3x\boldsymbol{i} - y\boldsymbol{j} - 2z\boldsymbol{k}$

Using this information, evaluate the absolute value of the line integral

$$\int_{\partial S} \boldsymbol{F} \cdot d\boldsymbol{s}.$$

Solution. Here we will use Stokes' theorem. Note that since S lies in the plane $y = 3$, the unit normal vector $\boldsymbol{n}$ at every point of S is the vector $\boldsymbol{n} = \boldsymbol{j}$ (or $-\boldsymbol{j}$ if we want to point the other direction). We compute

$$\begin{aligned}
\int_{\partial S} \boldsymbol{F} \cdot d\boldsymbol{s} &= \iint_S \text{curl}(\boldsymbol{F}) \cdot d\boldsymbol{S} \quad \text{Stokes' theorem,} \\
&= \iint_S \text{curl}(\boldsymbol{F}) \cdot \boldsymbol{j}\, dS \quad \text{since } \boldsymbol{n} = \boldsymbol{j}, \\
&= \iint_S -y\, dS \quad \text{from the given formula for curl}(\boldsymbol{F}), \\
&= \iint_S -3\, dS \quad \text{since } y = 3 \text{ for every point of } S, \\
&= -3 \iint_S 1\, dS \\
&= -3\text{Area}(S) \\
&= -51 \quad \text{since we are told that } S \text{ has area } 17.
\end{aligned}$$

If we used the other normal, we'd get 51, but in any case, the absolute value of the integral is $\boxed{51}$.

Problem 8. (10 points) Let $f(x, y) = \sqrt{x^4 + y^4 + 7}$. For any $a > 0$, let R_a be the rectangle

$$R_a = [-a, a] \times [-a, a].$$

Calculate

$$\lim_{a \to 0} \frac{1}{a^2} \iint_{R_a} f(x, y)\, dx\, dy.$$

Be sure to explain how you got your answer.

Solution. When a is very small, the value of the integral $\iint_{R_a} f\,dA$ is very close to $f(0,0)$ multiplied by the area of R_a. So

$$\begin{aligned}\lim_{a\to 0}\frac{1}{a^2}\iint_{R_a} f\,dA &= \lim_{a\to 0}\frac{1}{a^2}\cdot f(0,0)\cdot \text{Area}(R_a)\\ &= \lim_{a\to 0}\frac{1}{a^2}\cdot\sqrt{7}\cdot 4a^2\\ &= \boxed{4\sqrt{7}}.\end{aligned}$$

If you want to be more formal, you can quote the mean value theorem for integrals, which says that

$$\iint_{R_a} f\,dA = f(x_a,y_a)\cdot \text{Area}(R_a)$$

for some point (x_a,y_a) in R_a. Hence

$$\lim_{a\to 0}\frac{1}{a^2}\iint_{R_a} f\,dA = \lim_{a\to 0}\frac{1}{a^2}\cdot f(x_a,y_a)\cdot 4a^2 = 4\lim_{a\to 0} f(x_a,y_a) = 4f(0,0),$$

where the last equality comes from the fact that f is continuous and the fact that as $a\to 0$, the square R_a shrinks down to the point $(0,0)$.

Problem 9. (10 points) Let S_R be the sphere of radius R centered at the origin, taken with outward pointing normal. Let $\boldsymbol{F}$ be the vector field

$$\boldsymbol{F}(x,y,z) = x^3\boldsymbol{i} + z^3\boldsymbol{j} + y^3\boldsymbol{k}.$$

Use the Divergence Theorem to compute

$$\iint_{S_R} \boldsymbol{F}\cdot d\boldsymbol{S}.$$

Solution. Let Ω_R be the solid ball of radius R centered at the origin, so S_R is its boundary. Then

$$\begin{aligned}\iint_{S_R} \boldsymbol{F}\cdot d\boldsymbol{S} &= \iint_{\partial\Omega_R} \boldsymbol{F}\cdot d\boldsymbol{S}\\ &= \iiint_{\Omega_R} \text{div}(\boldsymbol{F})\,dV \quad \text{by the Divergence Theoerm,}\\ &= \iiint_{\Omega_R} 3x^2\,dV \quad \text{since } \text{div}(\boldsymbol{F}) = 3x^2.\end{aligned}$$

Clearly the way to compute this integral is using spherical coordinates. So

$$\iiint_{\Omega_R} 3x^2\,dV = \int_0^{\pi}\int_0^{2\pi}\int_0^{R} 3(\rho\cos\theta\sin\phi)^2\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$
$$= \int_0^{\pi}\int_0^{2\pi}\int_0^{R} 3\rho^4(\cos^2\theta)(\sin^3\phi)\,d\rho\,d\theta\,d\phi.$$

So we have three integrals to do.

$$\int_0^R \rho^4\,d\rho = \frac{1}{5}R^5.$$

$$\int_0^{2\pi}\cos^2\theta\,d\theta = \int_0^{2\pi}\frac{1+\cos(2\theta)}{2}\,d\theta = \frac{\theta}{2}+\frac{\sin(2\theta)}{4}\bigg|_0^{2\pi} = \pi.$$

$$\int_0^{\pi}\sin^3\phi\,d\phi = \int_0^{\pi}(1-\cos^2\phi)\sin\phi\,d\phi = \cos\phi - \frac{1}{3}\cos^3\phi\bigg|_0^{\pi} = \frac{4}{3}.$$

This gives the value

$$\iiint_{\Omega_R} 3x^2\,dV = 3\cdot\frac{1}{5}R^5\cdot\pi\cdot\frac{4}{3} = \boxed{\frac{4\pi R^5}{5}}$$

Here's a cleverer way to do the integral using an idea that was described in one of the problem sets. By symmetry, we have

$$\iiint_{\Omega_R} 3x^2\,dV = \iiint_{\Omega_R} 3y^2\,dV = \iiint_{\Omega_R} 3z^2\,dV.$$

But adding them gives an integral that's easy to compute using spherical coordinates,

$$\iiint_{\Omega_R} 3x^2+3y^2+3z^2\,dV = \int_0^{\pi}\int_0^{2\pi}\int_0^{R} 3\rho^2\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

since $x^2+y^2+z^2=\rho^2$,

$$= \int_0^{\pi}\int_0^{2\pi}\int_0^{R} 3\rho^4\sin\phi\,d\rho\,d\theta\,d\phi$$
$$= 3\cdot\frac{1}{5}\rho^5\bigg|_0^R\cdot\theta\bigg|_0^{2\pi}\cdot(-\cos\phi)\bigg|_0^{\pi}$$
$$= 3\cdot\frac{R^5}{5}\cdot 2\pi\cdot 2$$
$$= \frac{12\pi R^5}{5}.$$

Hence

$$\iiint_{\Omega_R} 3x^2\,dV = \frac{1}{3}\iiint_{\Omega_R} 3x^2+3y^2+3z^2\,dV = \boxed{\frac{4\pi R^5}{5}}$$