Principal Theorems of Vector Analysis

Mathematics 350 — Silverman — Fall, 2014

Notation and Definitions

$f(x,y,z)$ a function;

$\boldsymbol{F}(x,y,z) = F_1(x,y,z)\boldsymbol{i} + F_2(x,y,z)\boldsymbol{j} + F_3(x,y,z)\boldsymbol{k}$ a vector field;

$$\operatorname{grad} f = \nabla f = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k};$$

$$\operatorname{curl} \boldsymbol{F} = \nabla \times \boldsymbol{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\boldsymbol{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\boldsymbol{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\boldsymbol{k};$$

$$\operatorname{div} \boldsymbol{F} = \nabla \cdot \boldsymbol{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Fundamental Theorem of Calculus for Line Integrals

Let $\boldsymbol{c} : [a,b] \to \mathbb{R}^3$ be a path. Then

$$\int_{\boldsymbol{c}} \nabla f \cdot d\boldsymbol{s} = f(\boldsymbol{c}(b)) - f(\boldsymbol{c}(a)).$$

Green's Theorem

Let D be a region in the plane, and let C be the curve which forms its boundary. Also let $\boldsymbol{F}(x,y) = P(x,y)\boldsymbol{i} + Q(x,y)\boldsymbol{j}$ be a vector field. Then

$$\int_{C^+} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

Stokes' Theorem

Let S be an oriented surface, and let ∂S be the oriented boundary of S. Then

$$\iint_S (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S} = \int_{\partial S} \boldsymbol{F} \cdot d\boldsymbol{s}.$$

Gauss' Divergence Theorem

Let Ω be a solid region in space, and let $\partial\Omega$ be the oriented surface that bounds Ω. Then

$$\iiint_\Omega (\nabla \cdot \boldsymbol{F})\,dV = \iint_{\partial\Omega} \boldsymbol{F} \cdot d\boldsymbol{S}.$$