Solutions to
Math 420 HW #12
Due April 30, 2010

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Exercises

39.1. (a) Compute the first 10 terms in the continued fractions of $\sqrt{3}$ and $\sqrt{5}$.

(b) Do the terms in the continued fraction of $\sqrt{3}$ appear to follow a repetitive pattern? If so, prove that they really do repeat.

Solution to Exercise 39.1.
(a)

$$\sqrt{3} = [1, 1, 2, 1, 2, 1, 2, 1, 2, 1, \cdots],$$
$$\sqrt{5} = [2, 4, 4, 4, 4, 4, 4, 4, 4, 4, \cdots].$$

(b) Yes, they certainly look repetitive. Let $\alpha = [1, 2, 1, 2, 1, 2, \ldots]$, so

$$[1, 1, 2, 1, 2, 1, 2, 1, 2, 1, \cdots] = 1 + \frac{1}{\alpha}.$$

Since the continued fraction for α is purely periodic, we have

$$\alpha = 1 + \cfrac{1}{2 + \cfrac{1}{\alpha}}.$$

Simplifying gives

$$\alpha = \frac{3\alpha + 1}{2\alpha + 1},$$

so

$$2\alpha^2 - 2\alpha - 1 = 0.$$

Solving gives

$$\alpha = \frac{2 + \sqrt{12}}{4} = \frac{1 + \sqrt{3}}{2}.$$

Then

$$[1, 1, 2, 1, 2, 1, 2, 1, 2, 1, \cdots] = 1 + \frac{1}{\alpha} = 1 + \frac{2}{1 + \sqrt{3}} = 1 + \frac{2(1 - \sqrt{3})}{-2} = \sqrt{3}.$$

39.5. The Continued Fraction Recursion Formula (Theorem 39.1 gives a procedure for generating two lists of numbers $p_0, p_1, p_2, p_3, \ldots$ and $q_0, q_1, q_2, q_3, \ldots$ from two initial values a_0 and a_1. The fraction p_n/q_n is then the n^{th} convergent to some number α. Prove that the fraction p_n/q_n is already in lowest terms, that is, prove that $\gcd(p_n, q_n) = 1$. (*Hint.* Use the Difference of Successive Convergents Theorem 39.2.)

Solution to Exercise 39.5.
The Difference of Successive Convergents Theorem says that $p_{n-1}q_n - p_nq_{n-1} = (-1)^n$. Thus any common factor of p_n and q_n would also divide $(-1)^n$. Therefore p_n and q_n have no common factors larger than 1.

39.6. We proved that successive convergents p_{n-1}/q_{n-1} and p_n/q_n satisfy

$$p_{n-1}q_n - p_n q_{n-1} = (-1)^n.$$

In this exercise you will figure out what happens if instead we take every other convergent.

(a) Compute the quantity

$$p_{n-2}q_n - p_n q_{n-2} \qquad (*)$$

for the convergents of the partial fraction $\sqrt{2} = [1, 2, 2, 2, 2, \ldots]$. Do this for $n = 2, 3, \ldots, 6$.

(b) Compute the quantity (∗) for $n = 2, 3, \ldots, 6$ for the convergents of the partial fraction

$$\pi = [3, 7, 15, 1, 292, 1, 1, 1, 2, \ldots].$$

(c) Using your results from (a) and (b) (and any other data that you want to collect), make a conjecture for the value of the quantity (∗) for a general continued fraction $[a_0, a_1, a_2, \ldots]$.

(d) Prove that your conjecture in (c) is correct. (*Hint*. The Continued Fraction Recursion Formula may be useful.)

Solution to Exercise 39.6.

(a)

$$\begin{aligned}
p_0q_2 - p_2q_0 &= 1 \cdot 5 - 7 \cdot 1 = -2\\
p_1q_3 - p_3q_1 &= 3 \cdot 12 - 17 \cdot 2 = 2\\
p_2q_4 - p_4q_2 &= 7 \cdot 29 - 41 \cdot 5 = -2\\
p_3q_5 - p_5q_3 &= 17 \cdot 70 - 99 \cdot 12 = 2\\
p_4q_6 - p_6q_4 &= 41 \cdot 169 - 239 \cdot 29 = -2\\
p_5q_7 - p_7q_5 &= 99 \cdot 408 - 577 \cdot 70 = 2\\
p_6q_8 - p_8q_6 &= 239 \cdot 985 - 1393 \cdot 169 = -2\\
p_7q_9 - p_9q_7 &= 577 \cdot 2378 - 3363 \cdot 408 = 2
\end{aligned}$$

(b)

$$\begin{aligned}
p_0q_2 - p_2q_0 &= 3 \cdot 106 - 333 \cdot 1 = -15\\
p_1q_3 - p_3q_1 &= 22 \cdot 113 - 355 \cdot 7 = 1\\
p_2q_4 - p_4q_2 &= 333 \cdot 33102 - 103993 \cdot 106 = -292\\
p_3q_5 - p_5q_3 &= 355 \cdot 33215 - 104348 \cdot 113 = 1\\
p_4q_6 - p_6q_4 &= 103993 \cdot 66317 - 208341 \cdot 33102 = -1\\
p_5q_7 - p_7q_5 &= 104348 \cdot 99532 - 312689 \cdot 33215 = 1\\
p_6q_8 - p_8q_6 &= 208341 \cdot 265381 - 833719 \cdot 66317 = -2\\
p_7q_9 - p_9q_7 &= 312689 \cdot 364913 - 1146408 \cdot 99532 = 1
\end{aligned}$$

(c) The continued fraction of π is $\pi = [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, \ldots]$. That 292 also appears as the value of $p_2q_4 - p_4q_2$. This suggests that

$$p_{n-2}q_n - p_nq_{n-2} = (-1)^{n+1}a_n,$$

and all the other data in (a) and (b) support this conjecture.
(d) It is actually very easy to prove the conjecture using the results that we proved in this chapter, in particular the Continued Fraction Recursion Formulas

$$p_n = a_np_{n-1} + p_{n-2} \qquad \text{and} \qquad q_n = a_nq_{n-1} + q_{n-2}.$$

Using these formulas, we compute

$$\begin{aligned} p_{n-2}q_n - p_nq_{n-2} &= (p_n - a_np_{n-1})q_n - p_n(q_n - a_nq_{n-1}) \\ &\qquad \text{from the Continued Fraction Recursion Formulas} \\ &= -a_np_{n-1}q_n + p_na_nq_{n-1} \quad \text{since the } p_nq_n \text{ terms cancel,} \\ &= -a_n(p_{n-1}q_n - p_nq_{n-1}) \\ &= -a_n \cdot (-1)^n \\ &\qquad \text{from the Difference of Successive Convergents Theorem} \\ &= (-1)^{n+1}a_n. \end{aligned}$$

39.7. The "simplest" continued fraction is the continued fraction $[1, 1, 1, \ldots]$ consisting entirely of 1's.
(a) Compute the first 10 convergents of $[1, 1, 1, \ldots]$.
(b) Do you recognize the numbers appearing in the numerators and denominators of the fractions that you computed in (a)? (If not, look back at Chapter 37.)
(c) What is the exact value of the limit

$$\lim_{n\to\infty} \frac{p_n}{q_n}$$

of the convergents for the continued fraction $[1, 1, 1, \ldots]$?

Solution to Exercise 39.7.
The convergents p_n/q_n are ratios of Fibonacci numbers, as is easily proven by induction. The limiting value is the golden ratio $\frac{1+\sqrt{5}}{2}$.

40.1. Find the value of each of the following periodic continued fractions. Express your answer in the form $\frac{r+s\sqrt{D}}{t}$, where r, s, t, D are integers, just as we did in the text when we computed the value of $[1, 2, 3, \overline{4, 5}\,]$ to be $\frac{80-\sqrt{30}}{52}$.
(a) $[\,\overline{1, 2, 3}\,] = [1, 2, 3, 1, 2, 3, 1, 2, 3, 1, 2, 3, \ldots]$
(b) $[1, 1, \overline{2, 3}\,] = [1, 1, 2, 3, 2, 3, 2, 3, 2, 3, 2, 3, 2, 3, 2, \ldots]$

<u>*Solution to Exercise* 40.1</u>.
(a) $[1, 2, 3, 1, 2, 3, 1, 2, 3, 1, 2, 3, \ldots] = \frac{4+\sqrt{37}}{7}$.
(b) $[1, 1, 2, 3, 2, 3, 2, 3, 2, 3, 2, 3, 2, 3, 2, \ldots] = \frac{8-\sqrt{15}}{7}$,
where $[2, 3, 2, 3, 2, 3, 2, 3, 2, 3, 2, 3, 2, \ldots] = \frac{3+\sqrt{15}}{3}$.

40.4. Proposition 40.1 describes the number with continued fraction expansion $[a, \overline{b}\,]$.
(a) Do a similar computation to find the number whose continued fraction expansion is $[a, \overline{b, c}\,]$.
(b) If you let $b = c$ in your formula, do you get the same result as described in Proposition 40.1? (If your answer is "No," then you made a mistake in (a)!)

<u>*Solution to Exercise* 40.4</u>.
(a) $[a, \overline{b, c}\,] = \dfrac{(2a - c)b + \sqrt{bc(bc + 4)}}{2b}$.
(b) Putting $b = c$ lets us bring b outside the square root, then a little algebra gives the formula in the Proposition.

40.5. Theorem 40.3 tells us that if the continued fraction of $\sqrt{D}$ has odd period we can find a solution to $x^2 - Dy^2 = -1$.
(a) Among the numbers $2 \le D \le 20$ with D not a perfect square, which $\sqrt{D}$ have odd period and which have even period. Do you see a pattern?
(b) Same question for $\sqrt{p}$ for primes $2 \le p \le 40$. (See Table 40.1.)
(c) Write down infinitely many positive integers D so that $\sqrt{D}$ has odd period. For each of your D values, give a solution to the equation $x^2 - Dy^2 = -1$. (*Hint*. Look at Proposition 40.1.)

<u>*Solution to Exercise* 40.5</u>.
Very little is known about which $\sqrt{D}$ have odd or even period, or equivalently, for which D the equation $x^2 - Dy^2 = -1$ has a solution.
(c) The proposition says that $\sqrt{a^2 + 1} = [a, \overline{2a}\,]$, so $\sqrt{a^2 + 1}$ has period 1, which is odd. The equation $x^2 - (a^2 + 1)y^2 = -1$ has the solution $(a, 1)$.