Name: ______________________________

Honors Linear Algebra — Math 540
— Silverman —
First Hour Exam — Thurs Feb 20, 2020

INSTRUCTIONS—Read Carefully

- Time: 50 minutes
- There are 4 problems.
- Write your name **legibly** at the top of the page.
- No calculators or other electronic devices are allowed. (You won't need them.)
- **Show all your work**. Partial credit will be given for substantial progress towards the solution. **No credit** will be given for answers with no explanation.

Problem	Value	Points
1	12	
2	12	
3	14	
4	12	
Total	50	

Problem 1. (12 points) Let V be vector space over $\mathbb{F}$. Complete the following **definitions**:
(a) The vectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n \in V$ are *linearly independent* if...
(b) The vectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n \in V$ *span* V if...
(c) The vectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n \in V$ are a *basis* of V if...
(d) Let U and W be subspaces of V. The *sum* $U + W$ is the subspace...
(e) The vector space V is *finite dimensional* if...
(f) If the vector space V is finite dimensional, then its *dimension* is...

Solution.
(a) The vectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ are *linearly independent* if the only way to get $c_1\boldsymbol{v}_1 + \cdots + c_n\boldsymbol{v}_n = \boldsymbol{0}$ is to take $c_1 = \cdots = c_n = 0$.
(b) The vectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ *span* V if every vector in V is a linear combination of $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$. In other words, given any $\boldsymbol{v} \in V$, there are scalars $c_1, \dots, c_n \in \mathbb{F}$ so that $\boldsymbol{v} = c_1\boldsymbol{v}_1 + \cdots + c_n\boldsymbol{v}_n$.
(c) The vectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ are a *basis* of V if they are linearly independent and span V. Equivalently (modulo a small result proven in class), if every vector $\boldsymbol{v} \in V$ can be written as a linear combination $\boldsymbol{v} = c_1\boldsymbol{v}_1 + \cdots + c_n\boldsymbol{v}_n$ in exactly one way.
(d) The *sum* $U + W$ is the subspace

$$U + W = \{\boldsymbol{u} + \boldsymbol{w} : \boldsymbol{u} \in U \text{ and } \boldsymbol{w} \in W\}.$$

(e) The vector space V is *finite dimensional* if it has a finite spanning set.
(f) If the vector space V is finite dimensional, then its *dimension* is the number of vectors in a basis for V. (We proved that every basis has the same number of vectors.)

Problem 2. (12 points) Let V be a vector space. Let $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m$ be a list of linearly independent vectors in V, and let $\boldsymbol{u}_1, \dots, \boldsymbol{u}_n$ be another list of linearly independent vectors in V. Suppose further that

$$\operatorname{Span}(\boldsymbol{v}_1, \dots, \boldsymbol{v}_m) \cap \operatorname{Span}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_n) = \{\boldsymbol{0}\}.$$

Prove that

$$\boldsymbol{v}_1, \dots, \boldsymbol{v}_m, \boldsymbol{u}_1, \dots, \boldsymbol{u}_n$$

is a linearly independent list of vectors.

Solution. Suppose that there are scalars $a_1, \dots, a_m, b_1, \dots, b_n \in \mathbb{F}$ such that

$$a_1\boldsymbol{v}_1 + \cdots + a_m\boldsymbol{v}_m + b_1\boldsymbol{u}_1 + \cdots + b_n\boldsymbol{u}_n = \boldsymbol{0}.$$

Goal: Show that all of the scalars are zero.

We can rewrite the above equation as

$$a_1\boldsymbol{v}_1 + \cdots + a_m\boldsymbol{v}_m = -b_1\boldsymbol{u}_1 - \cdots - b_n\boldsymbol{u}_n.$$

The vector

$$a_1\boldsymbol{v}_1 + \cdots + a_m\boldsymbol{v}_m \text{ is in } \operatorname{Span}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$$

by the definition of the span of a list of vectors. Similarly, the vector

$$-b_1\boldsymbol{u}_1 - \cdots - b_n\boldsymbol{u}_n \text{ is in } \operatorname{Span}(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n).$$

Since these two vectors are equal, they are in the intersection

$$\operatorname{Span}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m) \cap \operatorname{Span}(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n).$$

But we are given that this intersection consists of only the zero vector. Therefore

$$a_1\boldsymbol{v}_1 + \cdots + a_m\boldsymbol{v}_m = -b_1\boldsymbol{u}_1 - \cdots - b_n\boldsymbol{u}_n = \mathbf{0}.$$

Now the linear independence of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$, which we are also given, implies by definition that $a_1 = \cdots = a_m = 0$, and similarly, the linear independence of $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ implies that $b_1 = \cdots = b_n = 0$. This completes the proof that $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ is a linearly independent list of vectors.

Problem 3. (14 points) Let $\mathcal{P}_2(\mathbb{F})$ be the vector space of polynomials of degree at most 2 with coefficients in $\mathbb{F}$, and let

$$U = \big\{p(x) \in \mathcal{P}_2(\mathbb{F}) : p(1) = p(-1) = 0\big\}.$$

(a) Prove that U is a subspace of $\mathcal{P}_2(\mathbb{F})$.
(b) Fill in the boxes:

$$\dim \mathcal{P}_2(\mathbb{F}) = \boxed{} \qquad \dim U = \boxed{}.$$

Justify your answers by giving a basis for $\mathcal{P}_2(\mathbb{F})$ and a basis for U.

Solution. (a) Let $p(x) \in U$ and $q(x) \in U$, and let $c \in \mathbb{F}$. Then

$$(p+q)(1) = p(1) + q(1) = 0 + 0 = 0 \quad \text{and}$$
$$(cp)(1) = c\big(p(1)\big) = c \cdot 0 = 0,$$

and similarly,

$$(p+q)(-1) = p(-1) + q(-1) = 0 + 0 = 0 \quad \text{and}$$
$$(cp)(-1) = c\big(p(-1)\big) = c \cdot 0 = 0.$$

Therefore $p(x) + q(x) \in U$ and $cp(x) \in U$, which shows that U is a subspace.

(b)

$$\dim \mathcal{P}_2(\mathbb{F}) = \boxed{3} \qquad \text{and} \qquad \dim U = \boxed{1}.$$

The elements of $\mathcal{P}_2(\mathbb{F})$ have the form

$$ax^2 + bx + c \quad \text{with } a, b, c \in \mathbb{F},$$

so $\boxed{\{x^2, x, 1\} \text{ is a basis for } \mathcal{P}_2(\mathbb{F})}$.

A polynomial $p(x) = ax^2 + bx + c$ is in U if and only if $p(1) = 0$ and $p(-1) = 0$, so if and only if

$$a + b + c = 0 \quad \text{and} \quad a - b + c = 0.$$

Subtracting these equations give $2b = 0$, so $b = 0$. Then we also need $a + c = 0$, so $c = -a$. Hence U consists of the polynomials

$$ax^2 - a \quad \text{with } a \in \mathbb{F}.$$

In other words, U is the set of scalar multiples of $x^2 - 1$, so

$$\boxed{\{x^2 - 1\} \text{ is a basis for } U.}$$

(We're taking $\mathbb{F}$ to be $\mathbb{R}$ or $\mathbb{C}$, as usual. But if you take $\mathbb{F}$ to be the field $\{0, 1\}$ containing only two elements, which we discussed briefly, then the answer to this problem changes, and in fact U has dimenison 2, with basis $\{x^2 + 1, x + 1\}$.)

Problem 4. (12 points) Let V be a vector space, and let U and W be subspaces of V. Suppose that

$$\dim(V) = 6 \qquad \text{and} \qquad \dim(U) = \dim(W) = 4.$$

Prove that there exist two linearly independent vectors $\boldsymbol{v}_1, \boldsymbol{v}_2 \in U \cap W$.

Solution. We use the dimension formula

$$\dim(U) + \dim(W) = \dim(U + W) + \dim(U \cap W).$$

From this we can compute

$$\begin{aligned} \dim(U \cap W) &= \dim(U) + \dim(W) - \dim(U + W) \\ &= 8 - \dim(U + W) \quad && (\text{since } \dim(U) = \dim(W) = 4) \\ &\geq 8 - \dim(V) && (\text{since } U + W \subseteq V) \\ &\geq 8 - 6 && (\text{since } \dim(V) = 6) \\ &= 2. \end{aligned}$$

We have proved that $\dim(U \cap W) \geq 2$, so in particular a basis for $U \cap W$ contains at least 2 vectors. Since the vectors in a basis are linearly independent, this shows that $U \cap W$ contains (at least) 2 linearly independent vectors.