Additional Problems for Math 252

Professor Silverman — Spring 2007

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The problems labeled A.1, A.2,... are additional problems for Math 252, Commutative Algebra.

Disclaimer: I have tried to make these exercises accurate, but if there are minor mistakes, then part of the assignment is to find and fix the mistakes. (For example, if I ask you to prove that a map F is a homomorphism between groups, then you need to check that $F(\sigma\tau) = F(\sigma)F(\tau)$. But it might turn out that instead $F(\sigma\tau) = F(\tau)F(\sigma)$.)

Let

$$F(\mathbf{x}; s, t) = x_0 s^r + x_1 s^{r-1} t + x_2 s^{r-2} t^2 + \cdots + x_{r-1} s t^{r-1} + x_r t^r$$

be a homogeneous polynomial of degree r in the variables s and t whose coefficients are the "variables" $x_0, \ldots, x_r$.

As in class, we can define an action of $\sigma = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}_2(k)$ on the polynomial ring $k[x_0, \ldots, x_r]$ by first expanding

$$F(\mathbf{x}; \alpha s + \beta t, \gamma s + \delta t) = x'_0 s^r + x'_1 s^{r-1} t + x'_2 s^{r-2} t^2 + \cdots + x'_{r-1} s t^{r-1} + x'_r t^r$$

and then, for any $f \in k[x_0, \ldots, x_r]$, setting

$$(\sigma f)(x_0, \ldots, x_r) = f(x'_0, \ldots, x'_r).$$

A.1. Let $r = 2$, so for convenience we will write

$$F(x, y, z; s, t) = xs^2 + yst + zt^2.$$

(a) Let $\sigma = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{GL}_2(k)$ Find the 9 entries of the matrix M_σ expressing the relationship between x, y, z and x', y', z', i.e., M_σ is the matrix making the following equation true:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Note that the entries of M_σ are polynomial functions of the entries of σ.
(b) Prove that the map

$$\mathrm{GL}_2(k) \longrightarrow \mathrm{GL}_3(k), \qquad \sigma \longrightarrow M_\sigma$$

is a homomorphism.
(c) Calculate $\det(M_\sigma)$ and relate it to $\det(\sigma)$.
(d) Let $\Delta = y^2 - 4xz$. Prove that

$$\sigma(\Delta) = \Delta \quad \text{for all } \sigma \in \mathrm{SL}_2(k).$$

Thus Δ is in the ring of invariants $k[x, y, z]^{\mathrm{SL}_2(k)}$.
(e) Prove the following equality (this may be a hard problem):

$$k[x, y, z]^{\mathrm{SL}_2(k)} = k[\Delta].$$

A.2. Factor $F(\mathbf{x}; s, t)$ over the algebraic closure of $k(x_0, \ldots, x_r)$ as

$$F(\mathbf{x}; s, t) = x_0 \prod_{i=1}^{r} (s - y_i t)$$

and let

$$\Delta = x_0^{2r-2} \prod_{\substack{1 \le i,j \le r \\ i \ne j}} (y_i - y_j).$$

(a) Prove that

$$\Delta \in k[x_0, \ldots, x_r].$$

(*Hint.* Start with the case $x_0 = 1$, in which case $x_0, \ldots, x_r$ are the elementary symmetric functions of $y_0, \ldots, y_r$. Prove that Δ is invariant under the permutation action of the symmetric group on $y_0, \ldots, y_r$ and then invoke the theorem on symmetric functions that I stated in class.)

For any particular set of values $a_0, \ldots, a_r$ with $a_0 \ne 0$, the polynomial $F(\mathbf{a}; s, 1) \in k[s]$ is a polynomial of degree r. Then $\Delta(a_0, \ldots, a_r) = 0$ if and only if the polynomial $F(\mathbf{a}; s, 1)$ has a double root in $\bar{k}$. Thus Δ, which is called the *discriminant of* F, lets us check if F has a double root using purely rational operations on the coefficients of F, i.e., we don't actually have to compute any roots.

(b) Let $r = 2$. Compute $\Delta = \Delta(x_0, x_1, x_2)$ explicitly. (If you like algebraic manipulation, do the same for $r = 3$.)

(c) Let $\mathrm{SL}_2(k)$ act on $k[x_0, \ldots, x_r]$ as described above. Prove that

$$\sigma\Delta = \Delta \qquad \text{for all } \sigma \in \mathrm{SL}_2(k).$$

In other words, prove that Δ is in the ring of invariants,

$$\Delta(x_0, \ldots, x_r) \in k[x_0, \ldots, x_r]^{\mathrm{SL}_2(k)}.$$

A.3. Let R be a ring and let M be an R-module.

(a) Let I be an ideal of R. Prove that

$$M \otimes_R \frac{R}{I} \cong \frac{M}{IM}.$$

(b) Let $(N_t)_{t\in T}$ be a collection of R-modules. Prove that

$$M \otimes_R \left(\bigoplus_{t\in T} N_t \right) \cong \bigoplus_{t\in T} (M \otimes_R N_t).$$

(c) With notation as in (b), what can you say about the relationship between

$$M \otimes_R \left(\prod_{t\in T} N_t \right) \quad \text{and} \quad \prod_{t\in T} (M \otimes_R N_t)?$$

In particular, can you find an example where they are not isomorphic.
(d) Prove that

$$\operatorname{Hom}_R\bigl(M, \operatorname{Hom}_R(N,P)\bigr) \cong \operatorname{Hom}_R(M \otimes_R N, P).$$

(In the language of categories, this says that the functor $N \mapsto M \otimes_R N$ is *left adjoint* to the functor $N \mapsto \operatorname{Hom}_R(M,N)$.)

A.4. Let R be a ring, let $I \subset R$ be an ideal of R, and let

$$\hat{R} = \hat{R}_I = \lim_{\leftarrow} R/I^{n+1}$$

be the "completion" of R with respect to I. Also let $\hat{I} = I\hat{R}$. As in class, we define a valuation v_I on $\hat{R}$ by setting

$$v_I(\mu) = \begin{cases} \min\{n \geq 0 : \mu \notin \hat{I}^{n+1}\} & \text{if } \mu \neq 0, \\ \infty & \text{if } \mu = 0. \end{cases}$$

(Equivalently, if you want to view μ as a coherent sequence

$$\mu = (a_0, a_1, a_2, \dots) \text{ with } a_n \in R/I^{n+1},$$

then $v_I(\mu)$ is the smallest n such that $a_n \neq 0$.) We also defined an associated I-adic absolute value

$$|\mu|_I = 2^{-v_I(\mu)}.$$

(a) Prove that $|\cdot|_I$ is an absolute value satisfying the nonarchimedean (or ultrametric) triangle inequality. In other words, prove the following properties of the I-adic absolute value:

(1) $|\mu|_I \ge 0$.

(2) $|\mu|_I = 0$ if and only if $\mu = 0$.

(3) $|\mu_1\mu_2|_I = |\mu_1|_I \cdot |\mu_2|_I$.

(4) $|\mu_1 + \mu_2|_I \le \max\bigl\{|\mu_1|_I, |\mu_2|_I\bigr\}$.

(b) In any metric space X, a sequence of elements $x_1, x_2, \ldots \in X$ is called a *Cauchy sequence* if

$$\lim_{n \ge m \to \infty} (\text{distance from } x_n \text{ to } x_m) = 0.$$

X is a *complete metric space* if every Cauchy sequence in X converges to an element of X. (For example, $\mathbb{R}$ and $\mathbb{C}$ are complete, but $\mathbb{Q}$ is not.) We can make $\hat{R}_I$ into a metric space by setting

$$(\text{distance from } \mu_1 \text{ to } \mu_2) = |\mu_1 - \mu_2|_I.$$

Prove that $\hat{R}_I$ is a complete metric space with respect to this I-adic metric, i.e., prove that every Cauchy sequence in $\hat{R}_I$ converges to an element of $\hat{R}_I$.