DYNAMICS AND RATIONAL MAPS
A NEW SECTION FOR CHAPTER 7 OF
THE ARITHMETIC OF DYNAMICAL SYSTEMS

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Let
\[ \varphi = [f_0, \ldots, f_N] : \mathbb{P}^N \to \mathbb{P}^N \]
be a rational map, where \( f_0, \ldots, f_N \) are homogeneous polynomials of degree \( d \) with no common factors. Then \( \varphi \) is defined at all points not in the indeterminacy locus
\[ Z(\varphi) = \{ P \in \mathbb{P}^N : f_0(P) = \cdots = f_N(P) = 0 \}. \]
The map \( \varphi \) is said to be dominant if the image
\[ \varphi(\mathbb{P}^N \setminus Z(\varphi)) \]
is Zariski dense, i.e., does not lie in a proper algebraic subset of \( \mathbb{P}^N \). Alternatively, the map \( \varphi \) is dominant if the polynomials \( f_0, \ldots, f_N \) do not satisfy a non-trivial polynomial relation.

If \( \varphi : \mathbb{P}^N \to \mathbb{P}^N \) is a morphism, then it is not hard to check that
\[ \deg(\varphi^n) = (\deg \varphi)^n \quad \text{for all } n \geq 1, \]
but if \( \varphi \) is merely assumed to be a dominant rational map, then the degree of \( \varphi^n \) may be strictly smaller than \( (\deg \varphi)^n \). For example,
\[ \varphi : \mathbb{P}^2 \to \mathbb{P}^2, \quad \varphi([x, y, z]) = [xy, xz, z^2] \]
satisfies
\[ \varphi^2([x, y, z]) = [x^2yz, xyz^2, z^4] = [x^2y, xyz, z^3], \]
so
\[ \deg(\varphi) = 2 \quad \text{and} \quad \deg(\varphi^2) = 3. \]
More generally, we have the following elementary results.

**Proposition 1.** Let
\[ \varphi : \mathbb{P}^N \to \mathbb{P}^M \quad \text{and} \quad \psi : \mathbb{P}^M \to \mathbb{P}^L \]
be dominant rational maps defined over a field \( K \).

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(a) We always have
\[ \deg(\psi \circ \varphi) \leq (\deg \psi)(\deg \varphi). \]

(b) If we further assume that \( \varphi \) and \( \psi \) are morphisms, then
\[ \deg(\psi \circ \varphi) = (\deg \psi)(\deg \varphi). \]

Proof. Let \( d = \deg(\varphi) \) and \( e = \deg(\psi) \), and write
\[
\varphi = [f_0, \ldots, f_M] \quad \text{with} \quad f_i \in K[X_0, \ldots, X_N],
\]
\[
\psi = [g_0, \ldots, g_L] \quad \text{with} \quad g_i \in K[X_0, \ldots, X_M],
\]
with homogeneous polynomials \( f_i \) and \( g_i \) of degrees \( d \) and \( e \), respectively. Then
\[
\psi \circ \varphi(\mathbf{X}) = [g_0(f_0(\mathbf{X}), \ldots, f_M(\mathbf{X})), \ldots, g_L(f_0(\mathbf{X}), \ldots, f_M(\mathbf{X}))]
\]
is given by homogeneous polynomials of degree \( de \), so
\[ \deg(\psi \circ \varphi) \leq de, \tag{1} \]
with equality if and only if
\[ g_0(f_0(\mathbf{X}), \ldots, f_M(\mathbf{X})) = \cdots = g_L(f_0(\mathbf{X}), \ldots, f_M(\mathbf{X})) = 0 \]
has no solutions in \( \mathbb{P}^N(\bar{K}) \). This proves (a).

For (b), we suppose that \( \deg(\psi \circ \varphi) < de \) and derive a contradiction.
Our assumption implies that there is a point \( P \in \mathbb{P}^N(\bar{K}) \) such that
\[ g_0(f_0(P), \ldots, f_M(P)) = \cdots = g_L(f_0(P), \ldots, f_M(P)) = 0. \]
But for (b) we are assuming that \( \psi \) is a morphism, so this implies that
\[ f_0(P) = \cdots = f_M(P) = 0, \]
which contradicts the assumption in (b) that \( \varphi \) is a morphism. We conclude that (1) is an equality, which completes the proof of (b).

In this chapter we study geometric and arithmetic properties of iterates of dominant rational maps.

1. The Dynamical Degree of a Dominant Rational Map

Let
\[ \varphi : \mathbb{P}^N \to \mathbb{P}^N \]
be a dominant rational map. Proposition 1(a) and an easy induction show that
\[ \deg(\varphi^n) \leq (\deg \varphi)^n, \]
so the sequence of degrees \( \deg(\varphi^n) \) grows at most exponentially as a function of \( n \). This prompts us to look at the following limit, which gives a rough measure of the growth rate of \( \deg(\varphi^n) \).
**Definition.** The *dynamical degree*\(^1\) of a dominant rational map \(\varphi : \mathbb{P}^N \to \mathbb{P}^N\) is the quantity

\[
\delta_\varphi = \lim_{n \to \infty} \left( \text{deg}(\varphi^n) \right)^{1/n}.
\]  

(2)

We give an example, state a conjecture, and then prove that the limit defining \(\delta_\varphi\) always converges.

**Example 2.** We continue with the example

\(\varphi : \mathbb{P}^2 \to \mathbb{P}^2, \quad \varphi([x, y, z]) = [xy, xz, z^2]\).

An easy computation gives degrees of the first few iterates,

\[
\begin{array}{c|cccc}
 n & 1 & 2 & 3 & 4 & 5 \\
 \hline
 \text{deg}(\varphi^n) & 2 & 3 & 5 & 8 & 13 \\
\end{array}
\]

From this data one naturally guesses that \(#\text{deg}(\varphi^n) = F_{n+2}\), where \(F_n\) is the \(n\)'th Fibonacci number, and a quick induction verifies that this is true. Hence

\[
\delta_\varphi = \lim_{n \to \infty} F_{n+2}^{1/n} = \frac{1 + \sqrt{5}}{2}.
\]

Thus \(\delta_\varphi\), which \textit{a priori} is only a real number, turns out to be an algebraic integer.

**Conjecture 3.** (Bellon–Viallet [3]) \textit{Let \(\varphi : \mathbb{P}^N \to \mathbb{P}^N\) be a dominant rational map. Then \(\delta_\varphi\) is an algebraic integer, i.e., \(\delta_\varphi\) is the root of a monic polynomial in \(\mathbb{Z}[X]\).}

**Theorem 4.** The limit (2) defining the dynamical degree converges to a real number \(\delta_\varphi \geq 1\).

**Proof.** From Proposition 1(a) we see that

\[
\text{deg}(\varphi^{n+m}) \leq \text{deg}(\varphi^n) \cdot \text{deg}(\varphi^m) \quad \text{for all } n, m \geq 1.
\]

(3)

To ease notation, we let

\[d_n = \log \text{deg}(\varphi^n).\]

Then (3) becomes the convexity estimate

\[d_{n+m} \leq d_n + d_m \quad \text{for all } n, m \geq 1,
\]

and our goal is to prove that \(\frac{1}{n}d_n\) has a limit as \(n \to \infty\).

Applying the convexity estimate repeatedly, we find that

\[d_{nk+r} \leq kd_m + d_r \quad \text{for all } m, k \geq 1 \text{ and } r \geq 0.
\]

\[\text{More precisely, the } \delta_\varphi \text{ that we have defined is the first dynamical degree. Higher-order dynamical degrees are described in Exercise 4.}\]
We now fix an \( m \geq 1 \), and for any \( n \) we write
\[
n = mk + r \quad \text{with } 0 \leq r < m,
\]
where \( k = k(n) \) and \( r = r(n) \) depend on \( n \). We compute
\[
\frac{1}{n} d_n = \frac{1}{n} d_{mk+r} \\
\leq \frac{1}{n} (kd_m + dr) \quad \text{from (4),}
\]
\[
= \frac{1}{m + \frac{r(n)}{k(n)}} \cdot d_m + \frac{1}{n} \cdot d_{r(n)}.
\]
We take the limsup as \( n \to \infty \). Then \( k(n) \to \infty \), while \( r(n) \) takes on only finitely many values, so we find that
\[
\limsup_{n \to \infty} \frac{1}{n} d_n \leq \frac{1}{m} d_m.
\]
This inequality holds for all \( m \geq 1 \), so we can take the liminf as \( m \to \infty \) to obtain
\[
\limsup_{n \to \infty} \frac{1}{n} d_n \leq \liminf_{m \to \infty} \frac{1}{m} d_m.
\]
But it is an elementary fact that
\[
\limsup_{n \to \infty} a_n \geq \liminf_{n \to \infty} a_n
\]
for any sequence \( (a_n) \) of real numbers, with equality if and only if \( \lim_{n \to \infty} a_n \) exists. This completes the proof that the limit defining \( \delta_\varphi \) converges, and, since \( \deg(\varphi^n) \geq 1 \) for all \( n \geq 1 \), it is clear that \( \delta_\varphi \geq 1 \).  

2. The Arithmetic Degree of an Orbit

The degree of a dominant rational map \( \varphi \) is a measure of its geometric complexity, so the dynamical degree is an asymptotic measure of the geometric complexity of the iterates of \( \varphi \). In a similar fashion, we might ask how fast the arithmetic complexity of the points \( \varphi^n(P) \) in an orbit grows as \( n \to \infty \).

As we saw in Chapter 3, the arithmetic complexity of a point \( Q \in \mathbb{P}^N(\overline{\mathbb{Q}}) \) is measured by its height \( h(Q) \). If \( \varphi : \mathbb{P}^N \to \mathbb{P}^N \) is a morphism of degree \( d \geq 2 \), then the existence and properties of the canonical height (Section 3.4)
\[
\hat{h}_\varphi(P) = \lim_{n \to \infty} \frac{1}{d^n} h(\varphi^n(P))
\]
give an accurate estimate
\[ h(\varphi^n(P)) = \hat{h}_\varphi(\varphi^n(P)) + O(1) = d^n \hat{h}_\varphi(P) + O(1) \]
for the arithmetic complexity of \( \varphi^n(P) \) as \( n \to \infty \). But if \( \varphi \) is only a dominant rational map, then the growth of \( h(\varphi^n(P)) \) may be much more complicated; see for example Exercises 1 and 3.

There is also the issue that a dominant rational map \( \varphi : \mathbb{P}^N \to \mathbb{P}^N \) is not defined at every point, and we want to study \( \varphi \)-orbits.

**Definition.** Let \( \varphi : \mathbb{P}^N \to \mathbb{P}^N \) be a dominant rational map. Then we define
\[
\mathbb{P}^N(\mathbb{Q})_\varphi = \{ P \in \mathbb{P}^N(\mathbb{Q}) : \varphi^n(P) \text{ is defined for all } n \geq 0 \}
\]
to be the set of points whose complete \( \varphi \)-orbit is well-defined.

**Remark 5.** It is easy to construct interesting dominant rational maps for which \( \mathbb{P}^N(\mathbb{Q})_\varphi \) is large. For example, let \( f_1, \ldots, f_N \in \overline{\mathbb{Q}}[x_1, \ldots, x_N] \) be algebraically independent polynomials. Then the affine morphism
\[
\varphi : \mathbb{A}^N \to \mathbb{A}^N, \quad \varphi(P) = (f_1(P), \ldots, f_N(P))
\]
extends to a dominant rational map \( \bar{\varphi} : \mathbb{P}^N \to \mathbb{P}^N \) satisfying \( \mathbb{P}^N(\mathbb{Q})_{\bar{\varphi}} \supset \mathbb{A}^N(\mathbb{Q}) \). On the other hand, it is not at all clear in general that \( \mathbb{P}^N(\mathbb{Q})_\varphi \) is large, or even that it is non-empty!

**Theorem 6.** (Amerik [1]) Let \( \varphi : \mathbb{P}^N \to \mathbb{P}^N \) be a dominant rational map of degree \( d \geq 2 \). Then \( \mathbb{P}^N(\mathbb{Q})_\varphi \) contains a Zariski dense set of wandering points having disjoint orbits.

The next result tells us that \( h(\varphi^n(P)) \) grows no faster than exponentially.

**Proposition 7.** Let \( \varphi : \mathbb{P}^N \to \mathbb{P}^N \) be a rational map of degree \( d \geq 2 \). Then there is a constant \( C_\varphi \) such that
\[
h(\varphi^n(P)) \leq (\deg \varphi)^n (h(P) + C_\varphi) \quad \text{for all } P \in \mathbb{P}^N(\overline{\mathbb{Q}}) \text{ and all } n \geq 0.
\]

**Proof.** The proof of the upper bound in Theorem 3.11 only requires that \( \varphi \) be a rational map of degree \( d \), so it implies that there is a constant \( C_\varphi \) such that
\[
h(\varphi(Q)) \leq dh(Q) + C_\varphi
\]
for all points \( Q \in \mathbb{P}^N(\overline{\mathbb{Q}}) \) at which \( \varphi \) is well-defined. Applying this estimate repeatedly gives
\[
h(\varphi^n(P)) \leq d^n h(P) + \frac{d^n - 1}{d - 1} C_\varphi,
\]
which gives the desired result with constant \( C_\varphi/(d - 1) \). \( \square \)
The upper bound in Proposition 7 and the definition of the dynamical degree $\delta_\varphi$ suggest the following arithmetic analogue, where to easy notation, we will let

$$h^+(P) = \max \{ h(P), 1 \}.$$ 

**Definition.** Let $\varphi : \mathbb{P}^N \to \mathbb{P}^N$ be a dominant rational map, and let $P \in \mathbb{P}^N(\overline{\mathbb{Q}})_\varphi$. The arithmetic degree of the $\varphi$-orbit of $P$ is the limit

$$\alpha_{\varphi}(P) = \lim_{n \to \infty} h^+ (\varphi^n(P))^{1/n},$$

if the limit exists. (We remark that the only reason to use $h^+$ instead of $h$ is for the rare situation that $P$ is preperiodic and some point in its orbit has height 0.)

The arithmetic degree of $P$ may also be characterized in terms of the height counting function of the orbit of $P$.

**Proposition 8.** Let $\varphi : \mathbb{P}^N \to \mathbb{P}^N$ be a dominant rational map, let $P \in \mathbb{P}^N(\overline{\mathbb{Q}})_\varphi$ be a wandering point, and assume that the arithmetic degree $\alpha_{\varphi}(P)$ exists. Then

$$\lim_{B \to \infty} \frac{\# \{ n \geq 0 : h(\varphi^n(P)) \leq B \}}{\log B} = \frac{1}{\log \alpha_{\varphi}(P)}.$$ 

(If $\alpha_{\varphi}(P) = 1$, we interpret this to mean that the limit is infinity.)

**Proof.** Let $\epsilon > 0$, and choose an $n_0(\epsilon)$ so that

$$(1 - \epsilon) \alpha_f(P) \leq h(f^n(P))^{1/n} \leq (1 + \epsilon) \alpha_f(P) \quad \text{for all } n \geq n_0(\epsilon).$$

It follows that

$$\{ n \geq n_0(\epsilon) : (1 + \epsilon) \alpha_f(P) \leq B^{1/n} \} \subset \{ n \geq n_0(\epsilon) : h(f^n(P)) \leq B \}$$

and

$$\{ n \geq n_0(\epsilon) : h(f^n(P)) \leq B \} \subset \{ n \geq n_0(\epsilon) : (1 - \epsilon) \alpha_f(P) \leq B^{1/n} \}.$$

Counting the number of elements in these sets yields

$$\frac{\log B}{\log((1 + \epsilon) \alpha_f(P))} - n_0(\epsilon) - 1 \leq \# \{ n \geq 0 : h(f^n(P)) \leq B \}$$

and

$$\# \{ n \geq 0 : h(f^n(P)) \leq B \} \leq \frac{\log B}{\log((1 - \epsilon) \alpha_f(P))} + n_0(\epsilon) + 1.$$ 

Dividing by $\log B$ and letting $B \to \infty$ gives

$$\frac{1}{\log((1 + \epsilon) \alpha_f(P))} \leq \liminf_{B \to \infty} \frac{\# \{ n \geq 0 : h(f^n(P)) \leq B \}}{\log B}.$$
and
\[ \limsup_{B \to \infty} \frac{\# \{ n \geq 0 : h(f^n(P)) \leq B \}}{\log B} \leq \frac{1}{\log((1-\epsilon)\alpha_f(P))}. \]

Since \( \epsilon \) is arbitrary, and the liminf is less than or equal to the limsup, this completes the proof that
\[ \lim_{B \to \infty} \frac{\# \{ n \geq 0 : h(f^n(P)) \leq B \}}{\log B} = 1 \]
including the fact that if \( \alpha_f(P) = 1 \), then the limit is \( \infty \).

The dynamical degree \( \delta \varphi \) measures the geometric complexity of the iterates of \( \varphi \), while the arithmetic degree \( \alpha_{\varphi}(P) \) measures the arithmetic complexity when the iterates of \( \varphi \) are applied to \( P \). These quantities are related by a basic inequality, but since we do not know, in general, that \( \alpha_{\varphi}(P) \) exists, we first define a quantity that does exist.

**Definition.** Let \( \varphi : \mathbb{P}^N \to \mathbb{P}^N \) be a dominant rational map, and let \( P \in \mathbb{P}^N(\overline{\mathbb{Q}})_{\varphi} \). The upper arithmetic degree of the \( \varphi \)-orbit of \( P \) is the limit
\[ \bar{\alpha}_{\varphi}(P) = \limsup_{n \to \infty} \frac{h^+(\varphi^n(P))^{1/n}}{\log \alpha_f(P)}. \]

We note that Proposition 7 implies that \( \bar{\alpha}_{\varphi}(P) \leq \deg(\varphi) \).

**Theorem 9.** Let \( \varphi : \mathbb{P}^N \to \mathbb{P}^N \) be a dominant rational map and let \( P \in \mathbb{P}^N(\overline{\mathbb{Q}})_{\varphi} \). Then
\[ \bar{\alpha}_{\varphi}(P) \leq \delta \varphi. \]

**Proof.** We know from (the proof of) Theorem 3.11 that for all dominant rational maps \( \psi \) there is a constant \( C(\psi) \geq 1 \) so that
\[ h(\psi(P)) \leq (\deg \psi)h(P) + C(\psi) \]
for all points \( P \in \mathbb{P}^N(\overline{\mathbb{Q}}) \) at which \( \psi \) is defined. Applying this repeatedly gives an estimate a little stronger than
\[ h(\psi^k(P)) \leq (\deg \psi)^k \left( h(P) + C(\psi) \right). \quad (5) \]

We fix an integer \( m \) and for each \( n \), we write
\[ n = mk + r \quad \text{with} \quad 0 \leq r < m, \]
where \( k = k(n) \) and \( r = r(n) \) depend on \( n \). We compute
\[ h(\varphi^n(P)) = h(\underbrace{\varphi^m \circ \varphi^m \circ \cdots \circ \varphi^m}_{k \text{ copies}} \circ \varphi^r(P)) \]
\[ \leq (\deg(\varphi^m))^k \left( h(\varphi^r(P)) + C(\varphi^m) \right) \quad \text{from (5) with} \quad \psi = \varphi^m. \]
We take $n$’th roots and let $n \to \infty$ to obtain
\[
\bar{\alpha}_\varphi(P) = \limsup_{n \to \infty} h(\varphi^n(P))^{1/n}
\leq \limsup_{n \to \infty} \left( \deg(\varphi^m)^{k/n} \left( h(\varphi^r(P)) + C(\varphi^m) \right)^{1/n} \right)
= \left( \deg(\varphi^m)^{1/m} \right).
\]
The last equality comes from the fact that as $n \to \infty$, we have $k \to \infty$, while $r$ takes on only finitely many values. In particular,
\[
\lim_{n \to \infty} \frac{k}{n} = \lim_{n \to \infty} \frac{k}{mk + r} = \lim_{n \to \infty} \frac{1}{m + \frac{r}{k}} = \frac{1}{m}.
\]
Letting $m \to \infty$ and using the definition of dynamical degree gives the desired result $\bar{\alpha}_\varphi(P) \leq \delta_\varphi$.

Theorem 9 raises the question of whether $\alpha_\varphi(P)$ can attain its maximum value $\delta_\varphi$. Part of the following conjecture gives a sufficient condition.

**Conjecture 10.** (Kawaguchi–Silverman [5]) Let $\varphi : \mathbb{P}^N \to \mathbb{P}^N$ be a dominant rational map of degree at least 2, and let $P \in \mathbb{P}^N(\bar{\mathbb{Q}})_r$.

(a) The arithmetic degree $\alpha_\varphi(P)$ exists.
(b) The arithmetic degree $\alpha_\varphi(P)$ is an algebraic integer.
(c) The set of arithmetic degrees $\{ \alpha_\varphi(Q) : Q \in \mathbb{P}^N(\bar{\mathbb{Q}})_r \}$ is finite.
(d) If the orbit $\mathcal{O}_\varphi(P)$ is Zariski dense in $\mathbb{P}^N$, then $\alpha_\varphi(P) = \delta_\varphi$.

**Remark 11.** If $\varphi : \mathbb{P}^N \to \mathbb{P}^N$ is a morphism of degree $d \geq 2$, then it is easy to see that Conjecture 10 is true; see Exercise 5.

### 3. The Geometry and Dynamics of Monomial Maps

The simplest rational maps on $\mathbb{P}^1$ are undoubtedly the power maps $\varphi(x) = x^d$. Monomial maps are a type of higher dimensional analogue of the power maps. Their simple structure makes their dynamics easier to study than general rational maps, but they still exhibit sufficient complexity to be quite interesting.

**Definition.** A *monomial map* is a rational map $\varphi : \mathbb{P}^N \to \mathbb{P}^N$ in which each coordinate function of $\varphi$ is given by a single non-zero monomial. Dehomogenizing by setting $X_0 = 1$, a monomial map is a map of the form
\[
\varphi(x_1, \ldots, x_N) = (x_1^{a_1}, x_2^{a_2}, \ldots, x_N^{a_N}, x_1^{a_1} x_2^{a_2}, \ldots, x_N^{a_N}, \ldots, x_1^{a_{N1}} x_2^{a_{N2}}, \ldots, x_N^{a_{NN}}).
\]
We let

\[ A = A_\varphi = (a_{ij})_{1 \leq i,j \leq N} \]

be the matrix of exponents that defines \( \varphi_A \). Conversely, each matrix \( A \) with integer entries gives a monomial map \( \varphi_A \). It is not hard to see that \( \varphi_A \) is dominant if and only if \( \det(A) \neq 0 \); see Exercise 6.

**Remark 12.** We write \( G_m \) for the multiplicative group. Then the \( N \)-dimensional torus \( G_m^N \) is a group via coordinate-wise multiplication,

\[ P \cdot Q = (x_1, \ldots, x_N) \cdot (y_1, \ldots, y_N) = (x_1y_1, \ldots, x_Ny_N). \]

It is easy to see that a monomial map \( \varphi \) defines a homomorphism

\[ \varphi : G_m^N \rightarrow G_m^N. \]

Further, for any two matrices \( A \) and \( B \), we have

\[ \varphi_{AB}(P) = \varphi_A \circ \varphi_B(P) \quad \text{and} \quad \varphi_{A+B}(P) = \varphi_A(P) \cdot \varphi_B(P). \]

The dynamical degree of a monomial map \( \varphi_A \) is determined by the matrix \( A \), as described in the next result.

**Definition.** The spectral radius of a matrix \( A \) with complex entries is the quantity

\[ \rho(A) = \max \{ |\lambda| : \lambda \in \mathbb{C} \text{ is an eigenvalue for } A \}. \]

**Theorem 13.** (Hasselblatt–Propp [4]) Let \( \varphi_A \) be the monomial map associated to an integer matrix \( A \) with \( \det(A) \neq 0 \). Then

\[ \delta_{\varphi_A} = \rho(A). \]

In particular, \( \delta_{\varphi_A} \) is an algebraic integer.

### 4. The Arithmetic of Orbits of Monomial Maps

Conjecture 10 for general rational maps appears quite difficult, with part (d) probably the lying the deepest. In this section we sketch the proof of Conjecture 10 for monomial maps. However, in order to exhibit the main tools that go into the proof while avoiding introducing various linear algebra technicalities, we will restrict attention to monomial maps whose matrices satisfy certain conditions. We refer the reader to [7] for the proof for general monomial maps.

**Theorem 14.** Let \( \varphi_A \) be a monomial map and let \( P \in G_m^N(\mathbb{Q}) \). Then \( \alpha_\varphi(P) \) exists, and it is either 1, or else it is equal to one of the eigenvalues of \( \varphi_A \). In particular, Conjecture 10(a,b,c) is true.

**Proof.** See [7].
We now turn to part (d) of Conjecture 10. We set one piece of notation. For each (non-zero) vector \( \mathbf{e} \in \mathbb{Z}^N \), we define a homomorphism
\[
\psi_\mathbf{e} : \mathbb{G}_m^N(\mathbb{C}) \to \mathbb{C}^*, \quad \psi_\mathbf{e}(P) = x_1^{e_1}x_2^{e_2} \cdots x_N^{e_N}.
\]
(6)
We note that if \( \mathbf{e} \neq \mathbf{0} \), then the kernel of \( \psi_\mathbf{e} \) is a proper subgroup of \( \mathbb{G}_m^N \); in particular, \( \ker(\psi_\mathbf{e}) \) is a proper Zariski closed subset of \( \mathbb{G}_m^N \).

**Theorem 15.** Let \( \varphi_A \) be a monomial map with matrix \( A \).

(a) There exists a non-zero vector \( \mathbf{e} \in \mathbb{Z}^N \) such that
\[
\{ P \in \mathbb{G}_m^N(\bar{\mathbb{Q}}) : \alpha_{\varphi_A}(P) < \delta_{\varphi_A} \} \subseteq \{ P \in \mathbb{G}_m^N(\bar{\mathbb{Q}}) : \psi_\mathbf{e}(P) \text{ is a root of unity} \}.
\]
(b) Let \( P \in \mathbb{G}_m^N(\bar{\mathbb{Q}}) \) be a point whose orbit \( \mathcal{O}_{\varphi_A}(P) \) is Zariski dense in \( \mathbb{P}^N \). Then \( \alpha_{\varphi_A}(P) = \delta_{\varphi_A} \).

**Proof Sketch of a Special Case.** We set some convenient notation. For \( Q = (y_1, \ldots, y_N) \in \mathbb{G}_m^N(\bar{\mathbb{Q}}) \), we let
\[
\log \|Q\|_v = \begin{pmatrix} \log \|y_1\|_v \\ \log \|y_2\|_v \\ \vdots \\ \log \|y_N\|_v \end{pmatrix} \in \mathbb{R}^N.
\]
This gives the nice formula
\[
\log \|\varphi_{\!n}(P)\|_v = A^n \log \|P\|_v.
\]
(7)
And for a vector \( \mathbf{r} = (r_1, \ldots, r_k) \in \mathbb{R}^k \), we write
\[
\max \mathbf{r} = \max\{r_1, \ldots, r_k\} \quad \text{and} \quad \max^+ \mathbf{r} = \max\{0, r_1, \ldots, r_k\}.
\]
We also let
\[
\varphi = \varphi_A \quad \text{and} \quad \delta = \delta_{\varphi_A}.
\]

Theorem 13 tells us that the dynamical degree of \( \varphi_A \) is equal to the spectral radius of \( A \). It follows that the eigenvalues of \( \delta^{-1}A \) have absolute value at most 1, so the set of powers
\[
\{ \delta^{-n}A^n : n \geq 1 \}
\]
is contained in a bounded, hence compact, subset of the set of \( N \times N \) matrices with complex coefficients. It thus has an accumulation point, so we can find a subsequence \( N \subset \mathbb{N} \) the limit so that
\[
\lim_{n \in N} \delta^{-n}A^n = B \quad \text{converges}.
\]
(8)
N.B. The coefficients of the matrix \( B \) will be real numbers, since \( \delta \) and the coefficients of \( A \) are real, but there is no reason that they should be algebraic numbers, much less rational numbers.
We consider a point $P \in G_N^m(\bar{Q})$ satisfying $\alpha_\varphi(P) < \delta_\varphi$. Let $\varepsilon = \frac{1}{2}(\delta_\varphi - \alpha_\varphi(P))$. Then $\varepsilon > 0$, so we can find an $n_0(\varepsilon)$ such that

$$h(\varphi^n(P))^{1/n} \leq \alpha_\varphi(P) + \varepsilon = \delta_\varphi - \varepsilon$$

for all $n \geq n_0(\varepsilon)$.

This allows us to compute

$$\limsup_{n \to \infty} \delta_\varphi^{-n}h(\varphi^n(P)) \leq \limsup_{n \to \infty} \delta_\varphi^{-n}(\delta_\varphi - \varepsilon)^n = 0.$$ 

But the quantity $\delta_\varphi^{-n}h(\varphi^n(P))$ is non-negative, so we conclude that

$$\lim_{n \to \infty} \delta_\varphi^{-n}h(\varphi^n(P)) = 0. \quad (9)$$

We now choose a number field $K$ so that $P \in G_N^m(K)$ and compute

$$0 = \lim_{n \to \infty} \delta_\varphi^{-n}h(\varphi^n_A(P)) \quad \text{from (9)},$$

$$= \lim_{n \to \infty} \delta_\varphi^{-n} \sum_{v \in M_K} \max^+ \log \|\varphi^n_A(P)\|_v \quad \text{definition of height},$$

$$= \lim_{n \to \infty} \sum_{v \in M_K} \max^+ \delta_\varphi^{-n} A^n \log \|P\|_v \quad \text{from (7)}. \quad (10)$$

The terms in the sum (10) are non-negative, so we must have

$$\lim_{n \to \infty} \max^+ \delta_\varphi^{-n} A^n \log \|P\|_v = 0 \quad \text{for all } v \in M_K.$$

In particular, the limit is still 0 if we restrict $n$ to be in the subsequence $N$, and then (8) tells us that

$$\max^+ B \log \|P\|_v = 0 \quad \text{for all } v \in M_K.$$

This formula says that for every $v \in M_K$, every coordinate of the vector $B \log \|P\|_v$ is non-positive.

On the other hand, the product formula gives

$$\sum_{v \in M_K} B \log \|P\|_v = B \left( \sum_{v \in M_K} \log \|P\|_v \right) = 0,$$

so when we sum over $v \in M_K$, every coordinate is zero. It follows that

$$B \log \|P\|_v = 0 \quad \text{for all } v \in M_K. \quad (11)$$

If the coefficients of $B$ were in $\mathbb{Q}$, then replacing $B$ by an integral multiple, we could assume that the coefficients are in $\mathbb{Z}$. Then we would have $\log \|\varphi_B(P)\|_v = 0$ for all $v \in M_K$, so Kronecker’s theorem (Theorem 3.8) would tell us that every coordinate of $\varphi_B(P)$ is a root of unity, which would more-or-less give us (a). Unfortunately, there is no reason why the coefficients of $B$ should be rational.
In order to simplify our exposition, we make the following assumption:

\begin{equation}
\text{The characteristic polynomial } \det(T - A) \text{ of } A \text{ has a unique largest eigenvalue.}
\end{equation}

In other words, we assume that there is one eigenvalue $\lambda_1$ satisfying $|\lambda_1| = \rho(A) = \delta$, and every other eigenvalue satisfies $|\lambda_i| < \rho(A)$. Writing $A$ in Jordan normal form, we have

$$A = C^{-1} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \Lambda \end{pmatrix} C,$$

where $\Lambda$ is an $(N - 1)$-by-$(N - 1)$ matrix consisting of Jordan blocks for the eigenvalues $\lambda_2, \ldots, \lambda_N$. We note that since $A$ has integral coefficients, the coefficient of $C$ and $C^{-1}$ are in $\mathbb{Q}$. This will be quite important.

Our assumption (12) implies that every eigenvalue of $\delta^{-1} \Lambda$ is strictly smaller than 1 in absolute value, so

$$\lim_{n \to \infty} (\delta^{-1} \Lambda)^n = 0.$$ 

This allows us to compute

$$B = \lim_{n \in \mathcal{N}} \delta^{-n} A^n = \lim_{n \in \mathcal{N}} C^{-1} \begin{pmatrix} (\delta^{-1} \lambda_1)^n & 0 \\ 0 & (\delta^{-1} \Lambda)^n \end{pmatrix} C$$

$$= C^{-1} \begin{pmatrix} \lim_{n \in \mathcal{N}} (\delta^{-1} \lambda_1)^n & 0 \\ 0 & 0 \end{pmatrix} C$$

$$= C^{-1} \begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix} C,$$

where the limit

$$\gamma = \lim_{n \in \mathcal{N}} (\delta^{-1} \lambda_1)^n$$

exists because the subsequence $\mathcal{N}$ was chosen so that the limit of $\delta^{-n} A^n$ exists, and is nonzero because it is a limit of quantities $\delta^{-1} \lambda_1$ having absolute value 1.\(^2\)

Substituting (13) into (11) gives

$$C^{-1} \begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix} C \log \|P\|_v = 0 \text{ for all } v \in M_K.$$

\(^2\)Our assumption implies that $\gamma$ is real, which combined with $|\gamma| = 1$ implies that $\gamma = \pm 1$. However, if $A$ has two or more largest eigenvalues, then the associated $\gamma$’s need not be real, or indeed even algebraic. So we will give the proof without assuming that $\gamma$ is real.
Writing
\[ C = (c_{ij}), \quad C^{-1} = (c'_{ij}), \quad \text{and} \quad P = (x_1, \ldots, x_N), \]
we find that
\[ \sum_{j=1}^{N} c'_{ij} \gamma c_{ij} \log \|x_j\|_v = 0 \quad \text{for all } 1 \leq i \leq N \text{ and all } v \in M_K. \]
We may cancel \( \gamma \), and since \( C \) is invertible, we can find an \( i \) such that \( c'_{i1} \neq 0 \). For this \( i \) we find that
\[ \sum_{j=1}^{N} c_{1j} \log \|x_j\|_v = 0 \quad \text{for all } v \in M_K. \]
We stress that all of the \( c_{1j} \) and all of the \( x_j \) are algebraic numbers.

To summarize, we have found a nonzero vector
\[ c \in \overline{\mathbb{Q}}^N \quad \text{such that} \quad c \cdot \log \|P\|_v = 0 \quad \text{for all } v \in M_K, \quad (14) \]
where \( \cdot \) denotes the standard dot product of two vectors. Further, the vector \( c \), which come from the decomposition (13) of the matrix \( B \), does not depend on the point \( P \).

**Claim 16.** Let \( P \in \mathbb{G}_m^N(\overline{\mathbb{Q}}) \). Then
\[ V_P = \{ \mathbf{a} \in \overline{\mathbb{Q}}^N : \mathbf{a} \cdot \log \|P\|_v = 0 \} \]
is a \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \)-invariant \( \overline{\mathbb{Q}} \)-vector subspace of \( \overline{\mathbb{Q}}^N \).

**Remark 17.** Claim 16 may appear to be innocuous, and for non-archimedean \( v \), it actually is quite easy to prove. But for archimedean \( v \), it is a very deep statement. To illustrate, we use it to prove that the number \( 2\sqrt{2} \) is transcendental, which is a famous Hilbert problem that was solved by Gelfond and Schneider. Suppose to the contrary that \( 2\sqrt{2} \) were algebraic. Then \( P = (2\sqrt{2}, 2) \in \mathbb{G}_m^2(\overline{\mathbb{Q}}) \). The relation
\[ \log(2\sqrt{2}) - \sqrt{2} \cdot \log(2) = 0 \]
shows that \( \mathbf{a} = (1, -\sqrt{2}) \in \overline{\mathbb{Q}}^2 \) is orthogonal to \( \log |P| \), where we use the usual archimedean absolute value on \( \mathbb{R} \). Let \( \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) satisfy \( \sigma(\sqrt{2}) = -\sqrt{2} \). Then the claim implies that \( \sigma(\mathbf{a}) = (1, \sqrt{2}) \) is also orthogonal to \( \log |P| \), so we find that
\[ \log(2\sqrt{2}) + \sqrt{2} \cdot \log(2) = 0. \]
But this implies that \( 2^{2\sqrt{2}} = 1 \), which gives the desired contradiction. Hence \( 2\sqrt{2} \) is not algebraic, so it is transcendental.
Proof of Claim 16. Suppose first that \( v \) is non-archimedean, say corresponding to a prime ideal whose norm is a power of \( p \). Then for every \( x \in K \), we have
\[
\|x\|_v = p^{r_v(x)} \quad \text{for some } r_v(x) \in \mathbb{Q}.
\]
Let \( a \in V_P \). Then
\[
0 = a \cdot \log \|P\|_v = \sum_{i=1}^N a_i \log \|x_i\|_v = \sum_{i=1}^N a_i \log p^{r_v(x_i)} = \sum_{i=1}^N a_i r_v(x_i) \log p.
\]
We can divide by \( \log p \) to obtain the relation
\[
\sum_{i=1}^N a_i r_v(x_i) = 0.
\]
Applying \( \sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \) gives
\[
\sum_{i=1}^N \sigma(a_i) r_v(x_i) = 0,
\]
since the \( r_v(x_i) \) are in \( \mathbb{Q} \). Multiplying by \( \log p \) and reversing the above calculation gives
\[
\sigma(a) \cdot \log \|P\|_v = 0,
\]
so \( \sigma(a) \in V_P \). This proves the desired \( \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \)-invariance of \( V_P \).

Now suppose that \( v \) is archimedean. As Remark 17 makes clear, the desired Galois invariance cannot be proven by an elementary argument. So we invoke a deep theorem on linear forms in logarithms.

Theorem 18. (Baker [2]) Fix an embedding \( \bar{\mathbb{Q}} \subset \mathbb{C} \), and choose a branch for the complex logarithm function \( \log : \mathbb{C}^* \to \mathbb{C} \). Let
\[
\alpha_1, \ldots, \alpha_t, \beta_1, \ldots, \beta_t \in \bar{\mathbb{Q}}^*
\]
satisfy
\[
\alpha_1 \log(\beta_1) + \alpha_2 \log(\beta_2) + \cdots + \alpha_t \log(\beta_t) = 0.
\]
Then there are rational numbers \( a_1, \ldots, a_t \in \mathbb{Q} \), not all 0, such that
\[
a_1 \log(\beta_1) + a_2 \log(\beta_2) + \cdots + a_t \log(\beta_t) = 0.
\]
In other words, if \( \beta_1, \ldots, \beta_t \) are algebraic numbers whose logarithms are linearly dependent over \( \bar{\mathbb{Q}} \), then their logarithms are linearly dependent over \( \mathbb{Q} \).
In the setting of Claim 16, Baker’s theorem tells us that if $V_P \neq 0$, then $V_P$ contains a non-zero vector $a_1 \in \mathbb{Q}^N$. The proof of Claim 16 proceeds by induction on the dimension of $V_P$. If $\text{dim}(V_P) = 1$, then we have

$$V_P = \{ \gamma a_1 : \gamma \in \bar{\mathbb{Q}} \},$$

from which it is clear that $V_P$ is Galois invariant.

We now suppose that $\text{dim}(V_P) = r \geq 2$, and we assume that Claim 16 is true for all $Q \in \mathbb{C}^m(\bar{\mathbb{Q}})$ for which $\text{dim} V_Q \leq r - 1$. Let

$$a_1, \ldots, a_r \in \bar{\mathbb{Q}}^N$$

be a $\bar{\mathbb{Q}}$-basis for $V_P$, where $a_1 \in \mathbb{Q}^N$ is the vector we obtained using Baker’s theorem, while $a_2, \ldots, a_N$ are only known to be in $\bar{\mathbb{Q}}^N$.

Permuting the coordinates of the $b_i$ and $P$, we may assume that the first coordinate of $a_1$ is non-zero. Then subtracting appropriate $\bar{\mathbb{Q}}$-multiples of $a_1$ from $a_2, \ldots, a_N$, we may assume that the first coordinates of $a_2, \ldots, a_N$ are zero. We now consider the point and vectors

$$P’ = (x_2, \ldots, x_N) \quad \text{and} \quad a’_i = (a_{i2}, a_{i3}, \ldots, a_{IN}) \quad \text{for } 2 \leq i \leq N.$$

We claim that $a’_2, \ldots, a’_N$ is a basis for $V_{P’}$. To see this, let $b \in V_{P’}$. Then $(0, b) \in V_P$, so

$$(0, b) = \sum_{i=1}^{N} c_i a_i = c_1 a_1 + \sum_{i=2}^{N} c_i (0, a’_i) \quad \text{for unique } c_i \in \bar{\mathbb{Q}}.$$

Looking at the first coordinate gives $0 = c_1 a_{11}$, so $c_1 = 0$, which proves that $b$ is in the span of $a’_2, \ldots, a’_N$, and the uniqueness of the $c_i$ implies that $a’_2, \ldots, a’_N$ are linearly independent. This proves that $\text{dim}(V_{P’}) = r - 1$, so by the induction hypothesis, it has a basis $b_2, \ldots, b_r \in \mathbb{Q}^N$. Then

$$a_1, (0, b_2), \ldots, (0, b_r) \quad \text{are in } \mathbb{Q}^N$$

and are a $\bar{\mathbb{Q}}$-basis for $V_P$, which shows that $V_P$ is $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-invariant. 

Resuming the proof of Theorem 15, we recall from (14) that we have constructed a nonzero $c \in \bar{\mathbb{Q}}^N$ such that $c \cdot \log \| P \|_v = 0$ for all $v \in M_K$. Further, the vector $c$ does not depend on the point $P$, so we may replace the field $K$ by a finite Galois extension so that $c \in K^N$. We consider the vector space

$$W = \sum_{\sigma \in \text{Gal}(K/\mathbb{Q})} \bar{\mathbb{Q}} \cdot \sigma(c) \subset \bar{\mathbb{Q}}^N,$$  \hspace{1cm} (15)
i.e., $W$ is the subspace spanned by the Galois conjugates of $c$. Claim 16 tells us that every conjugate $\sigma(c)$ annihilates $\log \|P\|_v$, so by linearity we see that

$$w \cdot \log \|P\|_v = 0 \quad \text{for all } w \in W \text{ and all } v \in M_K.$$ (16)

On the other hand, it is clear from the definition of $W$ that it is a $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-invariant subspace of $\bar{\mathbb{Q}}^N$. We now apply the following useful result.

**Lemma 19.** Let $W \subset \bar{\mathbb{Q}}^N$ be a $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-invariant vector subspace of $\bar{\mathbb{Q}}^N$. Then $W$ has a basis consisting of vectors in $\mathbb{Q}^N$. Equivalently, there is a vector space $W_0 \subset \mathbb{Q}^N$ such that $W = W_0 \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}$.

**Proof.** There are various proofs of this lemma; see for example [6, Lemma II.5.8.1 and Exercise 2.12].

Applying Lemma 19 to the vector space $W$ described in (15), we deduce in particular that there is a nonzero vector $e \in W \cap \mathbb{Z}^N$. (Note we are using the fact that $c \neq 0$, so $W$ is not the zero vector space.) Multiplying this vector by a large integer, we have constructed a vector

$$0 \neq e \in W \cap \mathbb{Z}^N.$$

And (16) tells us that this non-zero integral vector satisfies

$$e \cdot \log \|P\|_v = 0 \quad \text{for all } v \in M_K.$$ (17)

We use $e$ to define a map $\psi_e$ as described by (6). This allows us to write (17) as

$$\log \|\psi_e(P)\|_v = 0 \quad \text{for all } v \in M_K.$$  

Equivalently, we have $|\psi_e(P)|_v = 1$ for all $v \in M_K$. It follows from Kronecker’s theorem (Theorem 3.8) that $\psi_e(P)$ is a root of unity. We have thus shown that

$$\{P \in \mathbb{G}_m^N(\bar{\mathbb{Q}}) : \alpha_{\varphi_A}(P) < \delta_{\varphi_A} \} \subset \{P \in \mathbb{G}_m^N(\bar{\mathbb{Q}}) : \psi_e(P) \text{ is a root of unity} \},$$

which completes the proof of Theorem 15(a).

In order to prove (b), we let $P \in \mathbb{G}_m^N(\bar{\mathbb{Q}})$ be a point satisfying $\alpha_{\varphi_A}(P) < \delta_{\varphi_A}$, and we will prove that the orbit $O_{\varphi}(P)$ is not Zariski dense. We know from (a) that $\psi_e(P)$ is a root of unity. As before, we choose a number field $K/\mathbb{Q}$ such that $P \in \mathbb{G}_m^N(K)$. We let

$$d = \text{number of roots of unity in } K.$$
Since \( \psi_e(P) \in K^* \) is a root of unity, we find that \( \psi_e(P)^d = 1 \). Using the identity \( \psi_e = \psi_{de} \), we have proven that

\[
\{ P \in \mathbb{G}_m^N(K) \text{ and } \alpha_{\varphi}(P) < \delta_{\varphi} \} \implies \psi_{de}(P) = 1. \tag{18}
\]

We next observe that for any \( i \geq 0 \), we have

\[
\alpha_{\varphi}(\varphi^i(P)) = \lim_{n \to \infty} h(\varphi^{n+i}(P))^{1/n} = \lim_{n \to \infty} \left(h(\varphi^n(P))^{1/n}\right)^{\frac{n}{n-i}} = \alpha_{\varphi}(P).
\]

In other words, every point in the orbit \( \varphi^i(P) \) has the same arithmetic degree. So our assumption that \( \alpha_{\varphi}(P) < \delta_{\varphi} \) implies that the same is true for every point in \( \varphi^i(P) \). Further, every point in \( \varphi^i(P) \) is in \( \mathbb{G}_m^N(K) \). Applying (18) to each point in the orbit, we conclude that

\[
\varphi^i(P) \subset \ker(\psi_{de}).
\]

Since \( \ker(\psi_{de}) \) is a proper subgroup of \( \mathbb{G}_m^N \), this completes the proof of (b).

**Remark 20.** As the proof suggests, the statement of Theorem 15 may be strengthened as follows. If \( \alpha_{\varphi}(P) < \delta_{\varphi} \), then \( \varphi^i(P) \) is contained in a proper algebraic subgroup of \( \mathbb{G}_m^N \).

5. **Dynamical Degrees and Arithmetic Degrees on Other Varieties**

6. **Exercises**

**Exercise 1.** Consider the monomial map \( \varphi(x, y) = (x^{-d}, y^{-3}) \).

(a) Prove that \( \deg(\varphi^n) = d^n \) if \( n \) is even, and \( \deg(\varphi^n) = 2d^n \) if \( n \) is odd.

(b) Let \( \alpha, \beta \in \mathbb{Z} \) with \( \alpha \beta \neq 0 \). Prove that

\[
h(\varphi^n(P)) = \begin{cases} 
\log \max\{|x|, |y|\} & \text{if } n \text{ is even}, \\
\log(|xy|) & \text{if } n \text{ is odd}.
\end{cases}
\]

Deduce that \( \lim d^{-n}h(\varphi^n(P)) \) does not exist (unless \( |x| \) or \( |y| \) equals 1).

**Exercise 2.** Consider the rational map \( \varphi : \mathbb{P}^N \to \mathbb{P}^N \) defined by

\[
\varphi([X_0, \ldots, X_N]) = [X_0^{d+1}, X_1 X_2, X_2^d X_3, \ldots, X_{N-1}^d X_N, X_0 X_N^d].
\]

(This is a monomial map.)

(a) Prove that the limit

\[
\lim_{n \to \infty} \frac{\deg(\varphi^n)}{d^n - n^{N-1}}
\]

exists and is non-zero.

(b) Use (a) to compute the dynamical degree \( \delta_{\varphi} \).
(c) For each integer $0 \leq \ell \leq N - 1$, find an example of a monomial map $\varphi : \mathbb{P}^N \to \mathbb{P}^N$ such that the limit of $\frac{\text{deg}(\varphi^n)}{(d^n \cdot n^\ell)}$ exists and is non-zero.

**Exercise 3.** Let $\varphi : \mathbb{P}^3 \to \mathbb{P}^3$ be the map 
$$\varphi([X_0, X_1, X_2, X_3]) = [X_0^2, X_1(X_2 + X_3), X_0(X_2 + X_3), X_0X_3].$$

(a) Prove that $\text{deg}(\varphi^n) = n + 1$, so in particular $\delta_{\varphi} = 1$.
(b) Prove that
$$\lim_{n \to \infty} \frac{h(\varphi^n(1, 0, 1))}{n \log n} = 1.$$ Thus the growth rate of $h(\varphi^n(P))$ can have a logarithmic term in it.
(c) Open Questions: Does there exist a rational map $\varphi$ such that $h(\varphi^n(P))$ has a growth rate of the form $n^{\ell}(\log n)^k$ with $\ell \geq 1$ and $k \geq 2$? Can $h(\varphi^n(P))$ have a growth rate of the form $\delta_{\varphi} n^{\ell}(\log n)^k$ with $\ell \geq 0$ and $k \geq 1$?

**Exercise 4.** In this exercise we define higher order dynamical degrees. Let $\varphi : \mathbb{P}^N \to \mathbb{P}^N$ be a dominant rational map. Then for most linear subspaces $L \subset \mathbb{P}^N$ of codimension $i$, the inverse image $\varphi^{-1}(L)$ is a well-defined codimension $i$ subvariety of $\mathbb{P}^N$. The degree of $\varphi^{-1}(L)$, which by definition is the number of points in the intersection of $\varphi^{-1}(L)$ with a generic linear subspace of dimension $i$, does not depend on $L$. Then for each $1 \leq i \leq N$, the $i$’th dynamical degree is defined the quantity 
$$\delta_i(\varphi) = \lim_{n \to \infty} \left( \frac{\text{deg}((f^n)^{-1}(L))}{n^{1/n}} \right).$$

(a) Let $\varphi(X_0, \ldots, X_N) = [X_0^d, \ldots, X_N^d]$. Prove that $\delta_i(\varphi) = d^i$.
(b) Let $\varphi$ be the map in Exercise 2. Compute the higher dynamical degrees $\delta_i(\varphi)$ for $1 \leq i \leq N$.

**Exercise 5.** Let $\varphi : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism of degree $d \geq 2$, and let $P \in \mathbb{P}^N(\mathbb{Q})$. Prove that $\alpha_\varphi(P)$ exists and satisfies 
$$\alpha_\varphi(P) = \begin{cases} 
  d & \text{if } P \text{ is wandering,} \\
  1 & \text{if } P \text{ is preperiodic.} 
\end{cases}$$
Conclude that Conjecture 10 is true for morphisms.

**Exercise 6.** Let $A$ be a matrix with integer entries and let $\varphi_A$ be the associated monomial map. Prove that $\varphi_A$ is dominant if and only if $\text{det}(A) \neq 0$.

**Exercise 7.** Let $A$ and $B$ be matrices with associated monomial maps $\varphi_A$ and $\varphi_B$.

(a) Prove that $\varphi_{AB}(P) = \varphi_A \circ \varphi_B(P)$.
(b) Prove that $\varphi_{A+B}(P) = \varphi_A(P) \cdot \varphi_B(P)$. 
**Exercise 8.** Let $\varphi_A$ be the monomial map associated to an integer matrix $A$ with $\det(A) \neq 0$, and let $\rho(B)$ denote the spectral radius of the matrix $B$.

(a) Prove that $\rho(A) \geq 1$.
(b) Prove that the following are equivalent:
   (i) $\rho(A) = 1$.
   (ii) All of the eigenvalues of $A$ are roots of unity.
   (iii) There are positive integers $m$ and $n$ such that $(A^m - I)^n = 0$.
(c) If none of the eigenvalues of $A$ are roots of unity, prove that the set of preperiodic points of $\varphi_A$ in $\mathbb{G}^N_m(\mathbb{C})$ is

$$\text{PrePer}(\varphi_A) = \mathbb{G}^N_m(\mathbb{Q})_{\text{tors}},$$

where the torsion subgroup $\mathbb{G}^N_m(\mathbb{Q})_{\text{tors}}$ is the set of points whose coordinates are roots of unity.

**Exercise 9.** Let $e \in \mathbb{Z}^N$ be a non-zero vector, and let $\psi_e : \mathbb{G}^N_m(\mathbb{C}) \rightarrow \mathbb{C}^*$ be the homomorphism defined in (6). Prove that $\ker(\psi_e)$ is a connected subgroup of $\mathbb{G}^N_m(\mathbb{C})$ if and only if $\gcd(e_1, \ldots, e_N) = 1$.

**Exercise 10.** This exercise sketches two proofs of Lemma 19.

(a) Let $w_1, \ldots, w_r$ be a $\mathbb{Q}$-basis for $W$. Fix a Galois extension $K/\mathbb{Q}$ containing all of the coordinates of all of the $w_i$. For each $\sigma \in \text{Gal}(K/\mathbb{Q})$, the assumption that $W$ is Galois invariant means that we can write each $\sigma(w_i)$ as a $\mathbb{Q}$-linear combination of the other $w_j$, say

$$\sigma(w_i) = \sum_{j=1}^r a_{ij}\sigma w_j \quad \text{with} \quad a_{ij} \in \mathbb{Q}.$$

Let $A = (a_{ij})_{1 \leq i,j \leq r}$. Verify that

$$A_{\sigma\tau} = \sigma(A_{\tau})A_{\sigma} \quad \text{for all} \quad \sigma, \tau \in \text{Gal}(K/\mathbb{Q}).$$

This means that the map

$$\text{Gal}(K/\mathbb{Q}) \rightarrow \text{GL}_r(K), \quad \sigma \mapsto A_{\sigma},$$

is a 1-cocycle. Use the fact (Hilbert Theorem 90) that

$$H^1(\text{Gal}(K/\mathbb{Q}), \text{GL}_r(K)) = 0$$

to deduce that there is a matrix $B \in \text{GL}_r(K)$ satisfying

$$\sigma(A) = \sigma(B)^{-1}B \quad \text{for all} \quad \sigma \in \text{Gal}(K/\mathbb{Q}).$$

Finally, verify that the vectors $Bw_1, \ldots, Bw_r$ are in $\mathbb{Q}^N$.

(b) Let

$$W_0 = \{w \in W : \sigma(w) = w \text{ for all } \sigma \in \text{Gal}(\mathbb{Q}/\mathbb{Q})\}.$$

Verify that $W_0$ is a $\mathbb{Q}$-vector space. Let $w \in W$, and let $K/\mathbb{Q}$ be a finite Galois extension containing the coordinates of $w$. Let $\{\alpha_1, \ldots, \alpha_n\}$ be
a basis for $K/\mathbb{Q}$, let $\text{Gal}(K/\mathbb{Q}) = \{\sigma_1, \ldots, \sigma_n\}$, and for each $1 \leq i \leq n$, let

$$w_i = \sum_{j=1}^{n} \sigma_j(\alpha_i w).$$

Prove that $w_i \in W_0$, and prove that $w$ is a $\mathbb{Q}$-linear combination of $w_1, \ldots, w_n$. (Use the fact that the matrix $\left(\sigma_j(\alpha_i)\right)$ is non-singular.) Deduce that a $\mathbb{Q}$-basis for $W_0$ is also a $\mathbb{Q}$-basis for $W$.

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