Moduli Spaces
for Dynamical Systems

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Rational Maps on Projective Space

Notation: We fix

$$n \geq 1 \quad \text{and} \quad d \geq 2.$$

Primary Object of Study: Dynamics of

$$\mathrm{Rat}_d^n := \left\{ \text{Rational maps } f : \mathbb{P}^n \longrightarrow \mathbb{P}^n \text{ of degree } d \right\}.$$

Dynamics is the study of **Iteration**:

$$f^{\circ k} := \underbrace{f \circ f \circ f \circ \cdots \circ f}_{k \text{ copies}}.$$

The **Orbit** of a point $\alpha \in \mathbb{P}^n$ is

$$\mathcal{O}_f(\alpha) := \{ f^{\circ k}(\alpha) : k = 0, 1, 2, \ldots \}.$$

The point α is **Preperiodic** if it has finite orbit.

A Soupçon of Motivation

Arithmetic Dynamics is the study of arithmetic properties of iteration of maps and their orbits. Many problems are inspired by analogy from arithmetic geometry. Some examples:

- Let $f(z) \in \mathbb{Q}(z)$. Only finitely many $\alpha \in \mathbb{Q}$ are preperiodic (Northcott, 1950). Is there a bound for the number of such points that *depends only on* $\deg(f)$? (Uniform Boundedness Conjecture)
- Let $f(z) \in \mathbb{Q}(z)$ and $\alpha \in \mathbb{Q}$. When can $\mathcal{O}_f(\alpha)$ contain infinitely many integers? (Dynamical analogue of Siegel's theorem)
- Let $f : \mathbb{P}^n \to \mathbb{P}^n$ and $V \subseteq \mathbb{P}^n$ and $\alpha \in \mathbb{P}^n(\mathbb{C})$. When is $\#\mathcal{O}_f(\alpha) \cap V = \infty$? When is $\#\operatorname{Per}(f) \cap V = \infty$? (Dynamical analogues of Bombieri-Lang and Manin-Mumford conjectures)

A Soupçon of Motivation

- In arithmetic geometry, rather than studying one variety, it is fruitful to look at the space of all varieties.
- More precisely, one studies the space of *isomorphism classes* of varieties having a specified structure, i.e., moduli spaces.
- The aim of this talk is to describe some of the moduli spaces that come up in dynamics, and to see how they are analogous, in some ways, to moduli spaces such as $X_1(N)$, $\mathcal{M}_g$, and $\mathcal{A}_g$ that are so important in arithmetic geometry.

Rational Maps on $\mathbb{P}^1$

We start with $n = 1$. Rational maps $\mathbb{P}^1 \to \mathbb{P}^1$ look like

$$f(z) = \frac{a_0 z^d + a_1 z^{d-1} + \cdots + a_d}{b_0 z^d + b_1 z^{d-1} + \cdots + b_d},$$

but we get the same map if multiply the numerator and denominator by any non-zero c. Hence

$$\mathrm{Rat}_d^1 = \big\{[a_0, \ldots, a_d, b_0, \ldots, b_d]\big\} = \mathbb{P}^{2d+1}.$$

Convention: For now we allow "degenerate" maps where top and bottom have a common factor. To get a map of exact degree d, we need a_0, b_0 not both zero and no common factor.

Example:

$$\mathrm{Rat}_2^1 = \left\{ \frac{a_0 z^2 + a_1 z + a_2}{b_0 z^2 + b_1 z + b_2} : [a_0, \ldots, b_2] \in \mathbb{P}^5 \right\}.$$

Identifying Rat_d^n with $\mathbb{P}^N$

In general we write

$$f(X_0,\ldots,X_n) = \bigl[f_1(X_0,\ldots,X_n),\ldots,f_n(X_0,\ldots,X_n)\bigr]$$

with

$$f_i \in K[X_0,\ldots,X_n] \quad \text{homogeneous of degree } d.$$

Then we identity f with the list of its coefficients:

$$f \leftrightarrow [\text{coeffs. of } f] \in \mathbb{P}^N \text{ with } N = (n+1)\binom{n+d}{d} - 1.$$

The exact value of $N = N_d^n$ is not important, except to note that it gets quite large as $d \to \infty$. For example,

$$N_d^1 = 2d+1, \qquad N_d^2 = \frac{3d^2+9d+4}{2}.$$

Equivalence of Dynamical Systems

Question: When are two maps $f, g : \mathbb{P}^n \to \mathbb{P}^n$

Dynamically Equivalent?

Answer: When they differ by a change of variables of $\mathbb{P}^n$, i.e., when there is an *automorphism* $L : \mathbb{P}^n \to \mathbb{P}^n$ so that

$$g = f^L := L^{-1} \circ f \circ L, \qquad \begin{array}{ccc} \mathbb{P}^n & \xrightarrow{g} & \mathbb{P}^n \\ \downarrow L & & \downarrow L \\ \mathbb{P}^n & \xrightarrow{f} & \mathbb{P}^n \end{array}$$

N.B. Conjugation by L commutes with composition:

$$(f^L)^{\circ k} = f^L \circ \cdots \circ f^L = (f \circ \cdots \circ f)^L = (f^{\circ k})^L.$$

That's why conjugation is the appropriate equivalence relation for dynamics.

Equivalence of Dynamical Systems

For example, if $n = 1$, then

$$L = \frac{\alpha z + \beta}{\gamma z + \delta} \in \mathrm{PGL}_2$$

is a linear fractional transformation, and in general

$$L \in \mathrm{Aut}(\mathbb{P}^n) \cong \mathrm{PGL}_{n+1}$$

is given by $n+1$ linear forms in $n+1$ variables.
The conjugation action of

$$\mathrm{Aut}(\mathbb{P}^n) = \mathrm{PGL}_{n+1} \quad \text{on} \quad \mathrm{Rat}_d^n \cong \mathbb{P}^N$$

gives a ***very complicated*** homomorphism

$$\mathrm{PGL}_{n+1} \longrightarrow \mathrm{PGL}_{N+1}\,.$$

Primary Goal: Understand the quotient space*

$$\boxed{\mathrm{Rat}_d^n \,/\, \mathrm{PGL}_{n+1}\text{-conjugation}}$$

* Really $\mathrm{Rat}_d^n \,/\, \mathrm{SL}_{n+1}$ for technical reasons. Will ignore!

An Example

We illustrate with the map

$$\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2 \longrightarrow \mathrm{PGL}_6 = \mathrm{Aut}(\mathrm{Rat}_2^1) \cong \mathbb{P}^5.$$

The conjugation action of the linear fractional transformation $L = \frac{\alpha z+\beta}{\gamma z+\delta}$ on the vector of coefficients of the quadratic map $f = \frac{a_0z^2+a_1z+a_2}{b_0z^2+b_1z+b_2}$ is given by the matrix

$$\begin{pmatrix}
\alpha^2\delta & \alpha\gamma\delta & \gamma^2\delta & -\alpha^2\beta & -\alpha\beta\gamma & -\beta\gamma^2 \\
2\alpha\beta\delta & \alpha\delta^2+\beta\gamma\delta & 2\gamma\delta^2 & -2\alpha\beta^2 & -\alpha\beta\delta-\beta^2\gamma & -2\beta\gamma\delta \\
\beta^2\delta & \beta\delta^2 & \delta^3 & -\beta^3 & -\beta^2\delta & -\beta\delta^2 \\
-\alpha^2\gamma & -\alpha\gamma^2 & -\gamma^3 & \alpha^3 & \alpha^2\gamma & \alpha\gamma^2 \\
-2\alpha\beta\gamma & -\alpha\gamma\delta-\beta\gamma^2 & -2\gamma^2\delta & 2\alpha^2\beta & \alpha^2\delta+\alpha\beta\gamma & 2\alpha\gamma\delta \\
-\gamma\beta^2 & -\beta\gamma\delta & -\gamma\delta^2 & \alpha\beta^2 & \alpha\beta\delta & \alpha\delta^2
\end{pmatrix}$$

Geometric Invariant Theory to the Rescue

The quotient space

$$\mathrm{Rat}_d^n/(\mathrm{PGL}_{n+1}\text{-conjugation}).$$

makes sense as a set, but algebraically and topologically, it's a mess!

Geometric Invariant Theory (GIT, Mumford et al.) explains what to do. GIT says that if we restrict to a "good" subset of Rat_d^n, then we get a "good" quotient.

Moduli Spaces of (Semi)Stable Rational Maps

Theorem. There exist subsets of *semi-stable* and *stable* points so that the quotients

$$\overline{\mathcal{M}}_d^n := (\mathrm{Rat}_d^n)^{\mathrm{ss}} / \mathrm{PGL}_{n+1},$$
$$\mathcal{M}_d^n := (\mathrm{Rat}_d^n)^{\mathrm{stab}} / \mathrm{PGL}_{n+1},$$

are "nice" algebraic varieties.

- $(\mathrm{Rat}_d^n)^{\mathrm{stab}} \subseteq (\mathrm{Rat}_d^n)^{\mathrm{ss}}$ are non-empty Zariski open subsets of Rat_d^n, i.e., they're pretty big.
- $\mathcal{M}_d^n$ and $\overline{\mathcal{M}}_d^n$ are varieties, with $\overline{\mathcal{M}}_d^n$ projective.
- Two maps $f, g \in (\mathrm{Rat}_d^n)^{\mathrm{stab}}$ have the same image in $\mathcal{M}_d^n$ if and only if $g = f^L$ for some $L \in \mathrm{PGL}_{n+1}$.

Theorem. (Levy, Petsche–Szpiro–Tepper)

$$f \in \mathrm{Rat}_d^n \text{ a morphism} \implies \langle f \rangle \in \mathcal{M}_d^n.$$

Dynamical Moduli Space of Maps on $\mathbb{P}^1$

For $n = 1$, we know a fair amount.

Theorem. (Milnor over $\mathbb{C}$, JS as $\mathbb{Z}$-schemes)

$$\mathcal{M}_2^1 \cong \mathbb{A}^2 \quad \text{and} \quad \overline{\mathcal{M}}_2^1 \cong \mathbb{P}^2.$$

Milnor describes the isomorphism $\mathcal{M}_2^1 \cong \mathbb{A}^2$ very explicitly. I'll discuss this on the next slide.

Theorem. (Levy)

$$\mathcal{M}_d^1 \text{ is a rational variety.}$$

(It is clear that $\mathcal{M}_d^n$ is unirational.)

An Explicit Description of $\mathcal{M}_2^1$

- Degree 2 rational maps f have three fixed points:

$$f(\alpha_1) = \alpha_1, \quad f(\alpha_2) = \alpha_2, \quad f(\alpha_3) = \alpha_3.$$

- Compute the **multipliers**

$$\lambda_1 = f'(\alpha_1), \quad \lambda_2 = f'(\alpha_2), \quad \lambda_3 = f'(\alpha_3),$$

  and take the symmetric functions

$$\sigma_1(f) := \lambda_1 + \lambda_2 + \lambda_3, \quad \sigma_2(f) := \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3.$$

- **Fact**: $\sigma_1(f^L) = \sigma_1(f)$ and $\sigma_2(f^L) = \sigma_2(f)$.
- **Theorem**: The map

$$(\sigma_1, \sigma_2) : \mathcal{M}_2^1 \xrightarrow{\sim} \mathbb{A}^2 \quad \text{is an isomorphism.}$$

Theorem. (McMullen) If one uses enough symmetric functions of multipliers of periodic points, then one "usually" obtains a finite-to-one map $\mathcal{M}_d^1 \longrightarrow \mathbb{P}^*$.

"Special Points" in $\mathcal{M}_d^1$

Let $f \in \mathcal{M}_d^1$, and write $f(z) \in K(z)$. A **critical point** of f is a point α where $f'(\alpha) = 0$. A rational map of degree d has $2d - 2$ critical points (counted with multiplicities). The map is said to be **Post-Critically Finite** (PCF) if every critical point is preperiodic.

"Coincidentally" we also have $\dim \mathcal{M}_d^1 = 2d - 2$.

Theorem. (Thurston) There are only countably many PCF maps in $\mathcal{M}_d^1$. Properly formulated, they each appear with multiplicity one.

$$\textbf{Analogy}: \quad \underbrace{\text{PCF maps}}_{\text{dynamics}} \longleftrightarrow \underbrace{\text{CM abelian varieties}}_{\text{arithmetic geometry}}$$

There's been much work recently (Baker, DeMarco, . . .) on dynamical André–Oort type results, with PCF in place of CM.

Adding Level Structure

$$\textbf{Analogy}: \underbrace{\text{(Pre)periodic points}}_{\text{dynamics}} \longleftrightarrow \underbrace{\text{Torsion points}}_{\text{arithmetic geometry}}$$

We can add level structure to $\mathcal{M}_d^n$ by marking a periodic point:

$$\mathcal{M}_d^n(N) := \bigl\{(f,\alpha) : f \in \mathcal{M}_d^n \text{ and } \alpha \in \operatorname{Per}_N(f)\bigr\}.$$

Conjecture. $\mathcal{M}_d^n(N)$ is of general type for all sufficiently large N.

Theorem. (Blanc, Canci, Elkies)

- $\mathcal{M}_2^1(N)$ is rational for $N \le 5$.
- $\mathcal{M}_2^1(N)$ is of general type for $N = 6$.

The Automorphism Group of a Rational Map

General Principle: Given a category $\mathcal{C}$ and an object X, the group of automorphisms

$$\operatorname{Aut}_{\mathcal{C}}(X) = \{\text{isomorphisms } \phi : X \to X\}$$

is an interesting group.

Definition: The **automorphism group** of $f \in \operatorname{Rat}_d^n$ is the group

$$\operatorname{Aut}(f) = \left\{L \in \operatorname{PGL}_{n+1} : f^L = f\right\}.$$

In other words, $\operatorname{Aut}(f)$ is the set of fractional linear transformations that commute with f.

Observation:

$$\operatorname{Aut}(f^L) = L^{-1}\operatorname{Aut}(f)L,$$

so $\operatorname{Aut}(f)$ is well-defined (as an abstract group) for $f \in \mathcal{M}_d^n$.

The Automorphism Group of a Rational Map

Theorem. If f is a morphism, then $\mathrm{Aut}(f)$ is finite.

But note that $\mathrm{Aut}(f)$ may be infinite for dominant rational maps.

The theorem is not hard to prove, for example by using the fact that $\phi \in \mathrm{Aut}(f)$ must permute the points of period N. More precise results include:

$$f \in \mathcal{M}_d^n \Longrightarrow \#\mathrm{Aut}(f) \le C(n,d) \quad \text{(Levy)},$$
$$f \in \mathcal{M}_d^1 \Longrightarrow \#\mathrm{Aut}(f) \le \max\{60, 2d+2\},$$
$$f \in \mathcal{M}_d^2 \Longrightarrow \#\mathrm{Aut}(f) \le 6d^3 \quad \text{(de Faria–Hutz)},$$
$$f \in \mathcal{M}_2^2 \Longrightarrow \#\mathrm{Aut}(f) \le 21 \quad \text{(Manes–JS)},$$

The Automorphism Group of Maps of Degree 2 on $\mathbb{P}^1$

Proposition. Recall that there is a natural isomorphism $\mathcal{M}_2^1 \cong \mathbb{A}^2$. With this identification,

$$\{f \in \mathcal{M}_2^1 : \operatorname{Aut}(f) \text{ is non-trivial}\}$$

is a cuspidal cubic curve $\Gamma \subset \mathbb{A}^2$, and we have

$$\operatorname{Aut}(f) = \begin{cases} 1 & \text{if } f \notin \Gamma, \\ C_2 & \text{if } f = \Gamma \smallsetminus \text{cusp}, \\ \mathcal{S}_3 & \text{if } f = \text{cusp} \in \Gamma. \end{cases}$$

The non-cusp points of Γ are maps of the form

$$f(z) = bz + \frac{1}{z} \quad \text{with } b \neq 0.$$

The cusp is the map $f(z) = z^{-2}$.

Automorphisms and Twists

Just as with elliptic curves, the existence of non-trivial automorphisms leads to **twists**, that is, maps f and f' that are equivalent over the algebraic closure of K, but not over K itself. The dynamnics of distinct twists may exhibit different arithmetic properties.

The twists of f are classfied, up to K-equivalence, by

$$\operatorname{Ker}\Bigl(H^1(G_{\bar{K}/K}, \operatorname{Aut}(f)) \to H^1(G_{\bar{K}/K}, \operatorname{PGL}_{n+1}(\bar{K}))\Bigr).$$

Example: The map $f(z) = bz + z^{-1}$ on the last slide has $\operatorname{Aut}(f) = C_2$. Its quadratic twists are

$$f_c = bz + \frac{c}{z} \quad \text{with } c \in K^*.$$

Notice that

$$L(z) = \sqrt{c}\, z \quad \text{sends} \quad f_c^L(z) = f_1(z).$$

Maps in $\mathcal{M}_2^2$ with Large Automorphism Group

Joint work (in progress) with Michelle Manes

We classify semi-stable dominant rational maps

$$f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2 \quad \text{of degree 2 with } 3 \le \#\operatorname{Aut}(f) < \infty.$$

At this point we have a geometric classification.
Some interesting factoids gleaned from the classification:

- Let $f \in \overline{\mathcal{M}}_2^2$ with $\operatorname{Aut}(f)$ finite. Then $\operatorname{Aut}(f)$ is isomorphic to one of the following groups:

$$C_1,\ C_2,\ C_3,\ C_4,\ C_5,\ C_2^2,\ C_3 \rtimes C_2,\ C_3 \rtimes C_4,$$
$$C_2^2 \rtimes \mathcal{S}_3,\ C_4 \rtimes C_2,\ C_7 \rtimes C_3.$$

  Further, all of these groups occur.

- The map $f = [Y^2, X^2, Z^2]$ has $\operatorname{Aut}(f) \cong C_7 \rtimes C_3$.
- If $\operatorname{Aut}(f)$ contains a copy of either C_5 or $C_2 \times C_2$, then $f \in \overline{\mathcal{M}}_2^2 \smallsetminus \mathcal{M}_2^2$, so f is not a morphism.

Proof Idea

For a dominant rational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, let:

$$\begin{aligned} I(f) &= \text{Indeterminacy locus of } f \\ &= \{P \in \mathbb{P}^2 : f \text{ is not defined at } P\}, \\ \operatorname{Crit}(f) &= \text{Critical locus of } f \\ &= \text{Ramification locus of } f. \end{aligned}$$

Observation: Every $\phi \in \operatorname{Aut}(f)$ leaves $I(f)$ and $\operatorname{Crit}(f)$ invariant, i.e.,

$$\phi\bigl(I(f)\bigr) = I(f) \quad \text{and} \quad \phi\bigl(\operatorname{Crit}(f)\bigr) = \operatorname{Crit}(f).$$

Here $I(f)$ is either empty or a finite set of points, and $\operatorname{Crit}(f)$ is a cubic curve in $\mathbb{P}^2$.

Typical Examples

- If $I(f) = \{P_1, P_2\}$, then ϕ fixes or swaps P_1 or P_2.
- If $I(f) = L \cup C$, then $\phi(L) = L$ and $\phi(C) = C$.
- If $I(f) = L_1 \cup L_2 \cup L_3$, then ϕ permutes the lines.

Proof Idea

Combining information from

$$\phi\big(I(f)\big) = I(f), \quad \phi\big(\mathrm{Crit}(f)\big) = \mathrm{Crit}(f), \quad \phi \circ f = f \circ \phi$$

usually puts sufficient restrictions on ϕ to allow a case-by-case analysis.

The cases with

$$I(f) = \emptyset \quad \text{and} \quad \mathrm{Crit}(f) = \text{smooth cubic curve}$$

are interesting . For these we exploit the fact that $\phi\big(\mathrm{Crit}(f)\big) = \mathrm{Crit}(f)$ to note that ϕ preserves

(1) The 9 flex points of $\mathrm{Crit}(f)$, and (2) Lines.

This leads to 432 possible ways that ϕ can permute the 9 flex points, and each permutation π yields a 6-by-12 matrix $M_{\pi,\phi}$ such that f satisfies the constraint

$$\operatorname{rank} M_{\pi,\phi} \le 5.$$

A short computer program then checked all cases.

In Conclusion

I want to thank the organizers for inviting me to give this talk. But most of all, I want to say ....

$\mathcal{HAPPY}$ $\mathcal{BIRTHDAY}$ $\mathcal{RICHARD}$
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