Specialization Maps and Unlikely Intersections

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Specialization for Families of Abelian Varieties

An Example

We start with an example. Consider the family of elliptic curves and family of points,

$$E : y^2 = x^3 + x + T^2, \qquad P = (0, T).$$

We are interested in studying how frequently the *specialized point* $P_t = (0, t)$ is a point of finite order on the *specialized elliptic curve* E_t.

For example, setting $t = 0$, we find that

$$P_0 = (0, 0) \text{ is a point of order 2 on } E_0.$$

Similarly,

$$t = \frac{\sqrt[4]{\sqrt{5} - 1}}{2} \quad \text{makes } P_t \text{ a point of order 5 on } E_t.$$

The Specialization Map

In general, given an elliptic curve E defined over $K(T)$,

$$E : y^2 = x^3 + A(T)x + B(T),$$

each $t \in K$ defines a **Specialization Map**

$$\sigma_t : E\bigl(K(T)\bigr) \longrightarrow E_t(K).$$

Thus given a point $P = \bigl(x(T), y(T)\bigr) \in E\bigl(K(T)\bigr)$, we compute $\sigma_t(P)$ by evaluating at $T = t$:

$$\sigma_t(P) = \bigl(x(t), y(t)\bigr) \in E_t(K) : y^2 = x^3 + A(t)x + B(t).$$

It is natural to ask how frequently independent points in $E\bigl(K(T)\bigr)$ remain independent when specialized. Equivalently, how large is the **exceptional set**

$$\mathcal{E}(E, K) \stackrel{\text{def}}{=} \bigl\{t \in K : \sigma_t \text{ fails to be injective } \bigr\}?$$

Specialization Theorems

Theorem. Let $K/\mathbb{Q}$ be a number field and $E/K(T)$ an elliptic curve. Then

$$\mathcal{E}(E,K) \longrightarrow \begin{cases} \text{is small (density 0) (Néron, 1952),} \\ \text{is finite (JS, 1983).} \end{cases}$$

More generally, we look at one-parameter families of abelian varieties and consider specializations over $\bar{\mathbb{Q}}$.

Theorem. Let $T/\bar{\mathbb{Q}}$ be a curve and let $A \to T$ be a family of abelian varieties, defined over $\bar{\mathbb{Q}}$, with no constant part. Then

$$\mathcal{E}(A,\bar{\mathbb{Q}}) = \{t \in T(\bar{\mathbb{Q}}) : \sigma_t \text{ fails to be injective}\}$$

is a set of bounded height.

Heights and Specialization

The theorem is proven using a height specialization result. We fix an ample symmetric divisor D on A and consider three height functions:

$$\hat{h}_t : A_t(\bar{\mathbb{Q}}) \longrightarrow \mathbb{R}, \text{ canonical height wrt } D_t,$$
$$\hat{h} : A\big(\bar{\mathbb{Q}}(T)\big) \longrightarrow \mathbb{R}, \text{ canonical height wrt divisor } D,$$
$$h : T(\bar{\mathbb{Q}}) \longrightarrow \mathbb{R} \quad \text{height wrt degree 1 divisor.}$$

These heights are related by a limit formula:

Theorem. Let $P \in A\big(\bar{\mathbb{Q}}(T)\big)$. Then

$$\lim_{\substack{t \in T(\bar{\mathbb{Q}}) \\ h(t) \to \infty}} \frac{\hat{h}_t(P_t)}{h(t)} = \hat{h}(P).$$

The specialization theorem follows from nondegeneracy of $\hat{h}$ and $\hat{h}_t$.

Specialization in the Multiplicative Group

Specialization in the Multiplicative Group

Somewhat surprisingly, specialization results on elliptic curves and abelian varieties preceded study of the analogous question for the multiplicative group.

We again start with an example.

For which $t \in \mathbb{Q}$ are t and $t-2$ multiplicatively dependent?

If either t or $t-2$ equals 1 or -1, they are dependent, so that's $t \in \{-1, 1, 3\}$. Are there other $t \in \mathbb{Q}$?

$$t \text{ and } t-2 \text{ are multiplicatively dependent} \iff t \in \{-2, -1, 1, 3, 4\}.$$

Of course, if we allow $t \in \bar{\mathbb{Q}}$, there are many exceptional values. Indeed, each equation $t^n(t-2)^m = 1$ with $n, m \in \mathbb{Z}$, $(m, n) \neq (0, 0)$ gives finitely many $t \in \bar{\mathbb{Q}}$ such that t and $t-2$ are multiplicatively dependent.

Specialization in the Multiplicative Group

Let

$$f_1, \ldots, f_r \in \bar{\mathbb{Q}}(T)^*$$

be rational functions that are multiplicatively independent modulo $\bar{\mathbb{Q}}^*$. The associated **exceptional set** is

$$\mathcal{E}(f_1, \ldots, f_r) = \left\{ t \in \bar{\mathbb{Q}} : \begin{array}{l} f_1(t), \ldots, f_r(t) \text{ are} \\ \text{multiplicatively dependent} \end{array} \right\}.$$

Theorem. (Bombieri–Masser–Zannier, 1999)

$\mathcal{E}(f_1, \ldots, f_r)$ is a set of bounded height.

Heights and Specialization

The BMZ proof relies on height estimates, but note that the height is not a positive definite form on $\mathbb{G}_m$.

Theorem. (BMZ) There exist $C_1, C_2 > 0$ such that for all $m_1, \ldots, m_r \in \mathbb{Z}$ and all $t \in \bar{\mathbb{Q}}$,

$$h\bigl(f_1(t)^{m_1} f_2(t)^{m_2} \cdots f_r(t)^{m_r}\bigr) \geq \Bigl(\max_{1 \leq i \leq r} |m_i|\Bigr)\bigl(C_1 h(t) - C_2\bigr).$$

Proof Idea. • $\deg\left(f_1^{m_1} \cdots f_r^{m_r}\right) \gg \max |m_i|$.

• A general theorem says that

$$h\bigl(f(t)\bigr) \geq (\deg f) h(t) - c(f),$$

but that's no good, since $c(f)$ depends on f. It requires an intricate argument to replace the constant $c(f)$ with $\deg(f) c(f_1, \ldots, f_r)$ when f is in the group generated by $f_1, \ldots, f_r$.

An Unlikely Intersection

We can reformulate the BMZ result as follows. Consider the map

$$F = (f_1, \ldots, f_r) : T \longrightarrow \mathbb{A}^r.$$

Since T is a curve, we have

$$f_1(t), \ldots, f_r(t) \text{ are multiplicatively dependent} \iff F(t) \text{ lies on some } \underbrace{X_1^{e_1} \cdots X_r^{e_r} = 1}_{\text{subgroup of } \mathbb{G}_m^r}.$$

Thus the exceptional set is the intersection

$$\mathcal{E}(f_1, \ldots, f_r) = \underbrace{\text{Image}(F) \cap \begin{pmatrix} \text{subgroups} \\ \text{of } \mathbb{G}_m^r \end{pmatrix}}_{\textbf{Unlikely Intersection}}.$$

The BMZ results says that this unlikley intersection is a set of bounded height.

Higher Dimensional Families

Higher Dimensional Families

Up to now we have been considering one-paramter families, i.e., $\dim T = 1$. Things become much more complicated when $\dim T \geq 2$. Let

$$f_1, \ldots, f_r \in \bar{\mathbb{Q}}(T_1, \ldots, T_n)^*$$

be multiplicatively independent rational functions of n variables. Each relation

$$f_1(t)^{e_1} \cdots f_r(t)^{e_r} = 1$$

cuts the dimension of the solution set by one, so each n *independent* relations gives a finite set of points.

Problem. Describe the *exceptional set*

$$\mathcal{E}(f_1, \ldots, f_r) = \left\{ t \in \bar{\mathbb{Q}}^n : \begin{array}{l} f_1(t), \ldots, f_r(t) \text{ satisfy} \\ n \text{ independent} \\ \text{multiplicative relations} \end{array} \right\}.$$

Higher Dimensional Unlikely Intersections

We again reformulate the problem by looking at the map

$$F = (f_1, \ldots, f_r) : \mathbb{A}^n \longrightarrow \mathbb{A}^r.$$

Then

$$\mathcal{E}(F) = (\text{Image of } F) \quad \cap \bigcup_{\substack{H \subset \mathbb{G}_m^r \\ \text{subgroup of codim } n}} H.$$

It is natural to guess that

$$\mathcal{E}(F) \overset{?}{\subset} \begin{pmatrix} \text{proper Zariski} \\ \text{closed set} \end{pmatrix} \cup \begin{pmatrix} \text{set of bounded} \\ \text{height} \end{pmatrix}.$$

This guess is correct for $r = n$, which is the case

$$\mathcal{E}(F) = \{t \in \bar{\mathbb{Q}}^n : f_1(t), \ldots, f_n(t) \text{ are roots of unity}\}.$$

However, in general it is **not correct**.

A Counterexample

This example was shown to me by David Masser.

$$n = 2, \quad r = 3, \quad f_1 = X, \quad f_2 = Y, \quad f_3 = X + Y.$$

For $u, v \in \mathbb{N}$, let $\beta \in \bar{\mathbb{Q}}$ be a root of

$$\beta^u + \beta^v = 1 \quad \text{and specialize} \quad X = \beta^u,\ Y = \beta^v.$$

Then the specialized values

$$f_1 = \beta^u, \qquad f_2 = \beta^v, \qquad f_3 = 1,$$

satisfy two independent relations

$$f_1^v f_2^u = 1 \qquad \text{and} \qquad f_3 = 1.$$

But for any B, the set

$$\bigcup_{u,v \in \mathbb{N}} \{(\beta^u, \beta^v) : \beta^u + \beta^v = 1,\ h(\beta^u) > B\}$$

is Zariski dense in $\mathbb{A}^2$, so $\mathcal{E}(f_1, f_2)$ is not in the union of a set of bounded height and a proper closed subset.

Anomalous Subvarieties and the Bounded Height Theorem

Anomalous Subvarieties

Definition. Let
$$Y \subset X \subset \mathbb{G}_m^r.$$

Y is **anomalous** (**for X**) if $\dim Y \geq 1$ and there exists a coset (translate of a subgroup) $K \subset \mathbb{G}_m^r$ satisfying

$$Y \subset K \quad \text{and} \quad \dim Y > \underbrace{\dim X + \dim K - r}_{\text{expected dimension of } X \cap K}.$$

Since $Y \subset X \cap K$, this means that X and K contain an **unlikey intersection**.

Definition. The **non-anomalous part** of X is

$$X^{\text{na}} = X \setminus (\text{all anomalous subvarieties}).$$

Example. $X = \{(x, y, x + y)\} \subset \mathbb{G}_m^3 \Longrightarrow X^{\text{na}} = \emptyset.$

Theorem. (BMZ) X^{na} is a Zariski open subset of X.

The Bounded Height Theorem

The following beautiful result was conjectured by Bombieri, Masser and Zannier (2007) and proven by Habegger.

Notation.

$$\mathcal{G}^{[d]} = \left(\begin{array}{l} \text{union of all algebraic subgroups} \\ \text{of } \mathbb{G}_m^r \text{ of codimension } d \end{array} \right)$$

Theorem. (Habegger 2009)

$$X^{\text{na}} \cap \mathcal{G}^{[\dim X]} \text{ is a set of bounded height.}$$

More generally, Habegger shows that the result is true for points that are "close to $\mathcal{G}^{[\dim X]}$."

Theorem. There exists an $\epsilon > 0$ such that

$$X^{\text{na}} \cap \left\{ xy : \begin{array}{l} x \in \mathcal{G}^{[\dim X]},\ y \in \mathbb{G}_m^r, \\ \text{and } h(y) \le \epsilon(h(x)+1) \end{array} \right\}$$

is a set of bounded height.

Sketch of Habegger's Proof

Let $n = \dim X$. Construct a set of algebraic quotient groups $\Gamma_1, \ldots, \Gamma_t$ of $\mathbb{G}_m^r$ such that every $P \in \mathbb{G}_m^r$ lying in a subgroup of codimension n is in the kernel of some $\mathbb{G}_m^r \to \Gamma_i$. It thus suffices to fix one

$$\psi : \mathbb{G}_m^r \longrightarrow \Gamma.$$

In order to compare the heights of P and $\psi(P)$, Habegger uses the following theorem of Siu.

Theorem. (Siu 1993) Let X be an irreducible projective variety of dimension $n \geq 1$ defined over $\mathbb{C}$. Let $\mathcal{L}$ and $\mathcal{M}$ be nef line bundles on X. If

$$c_1(\mathcal{L})^n[X] > n\big(c_1(\mathcal{L})^{n-1}c_1(\mathcal{M})[X]\big),$$

then there exists an integer $k \geq 1$ such that $(\mathcal{L} \otimes \mathcal{M}^{-1})^k$ has a nonzero global section.

Sketch of Habegger's Proof (continued)

Habegger uses Siu's theorem to construct an effective divisor D satisfying

$$h_D\bigl(\psi(P)\bigr) \ge c(\psi)h_D(P) \quad \text{for } P \in \bigl(X \smallsetminus |D|\bigr)(\bar{\mathbb{Q}}),$$

where

$$c(\psi) \ge 0 \quad \text{is given as an intersection number.}$$

In order to show that $c(\psi) > 0$, Harbegger proves that it extends to a continuous map

$$c : \operatorname{Hom}(\mathbb{G}_m^r, \Gamma) \otimes \mathbb{R} \longrightarrow \mathbb{R}$$

and uses properties of $X^{\mathrm{na}} \cap$ (subgroups) and a (special case of a) theorem of Ax on analytic subgroups of $\mathbb{G}_m^r(\mathbb{C})$ to prove positivity.

The proof also requires working on an appropriate compactification of the group law map $\mathbb{G}_m^r \times \mathbb{G}_m^r \to \mathbb{G}_m^r$.

The Case $r = n$

We sketch an elementary proof of Habegger's theorem when $r = n$. We thus have a rational map

$$F = (f_1, \ldots, f_n) : \mathbb{G}_m^n \longrightarrow \mathbb{G}_m^n.$$

If F is not dominant, a result of Laurent gives the desired result. Let

$$\phi : V \dashrightarrow W$$

be a rational map of varieties. The triangle inequality gives an elementary upper bound (on an open set)

$$h\bigl(\phi(t)\bigr) \ll h(t).$$

Proposition. Assume that $\dim V = \dim W$ and that ϕ is dominant. Then

$$h\bigl(\phi(t)\bigr) \gg h(t) \quad \text{on a nonempty open subset of } V.$$

The Case $r = n$ (continued)

Proof Sketch.

$$\dim V = \dim W \text{ and } \phi \text{ dominant} \quad \Longrightarrow \quad k(V)/\phi^* k(W) \text{ is a finite extension.}$$

Let $f \in k(V)$, so f is a root of

$$X^d + A_1 X^{d-1} + \cdots + A_d = 0, \qquad A_i \in \phi^* k(W).$$

There is an open subset $U \subset V$ so that for all $t \in U$,

$$f(t) \text{ is a root of } X^d + A_1(t) X^{d-1} + \cdots + A_d(t) = 0.$$

Standard estimates relating the height of the coefficients of a polynomial to its roots gives

$$h\big(f(t)\big) \le h\big([1, A_1(t), \ldots, A_d(t)]\big) + O(1). \qquad (*)$$

The Case $r = n$ (continued)

Write $A_i = \phi^* B_i$ and define

$$\alpha = [1, A_1, \ldots, A_d] : V \dashrightarrow \mathbb{P}^d,$$
$$\beta = [1, B_1, \ldots, B_d] : W \dashrightarrow \mathbb{P}^d.$$

Then $(*)$ says

$$h\bigl(f(t)\bigr) \le h\bigl(\alpha(t)\bigr) + O(1).$$

The elementary height estimate gives

$$h\bigl(\beta(x)\bigr) \ll h(x) + O(1).$$

$$\begin{aligned} h\bigl(f(t)\bigr) &\le h\bigl(\alpha(t)\bigr) + O(1) \\ &= h\bigl(\beta \circ \phi(t)\bigr) + O(1) \ \text{ since } \alpha = \beta \circ \phi, \\ &\ll h\bigl(\phi(t)\bigr) + O(1). \end{aligned}$$

Now apply this to each of the coordinate functions of some embedding $V \subset \mathbb{P}^N$. $\square$

“Very” Unlikely Intersections

Increasing the Codimension

The codimension condition used to define unlikely intersections between subvarieties and subgroups is set up so that each individual $X \cap H$ is finite. But since there may be infinitely many H, the exceptional set consisting of all unlikely intersections

$$\mathcal{E} = X \cap \bigcup_{H} H$$

is generally infinite. In this case, one hopes to prove that the points in $\mathcal{E}$ in a geometrically described subset have bounded height.

If we increase the codimension condition by one, then most individual intersections $X \cap H$ should be empty, so we might expect the full exceptional set to be finite.

"Very" Unlikely Intersections

Conjectures for **Very Unlikely Intersections** of this sort were originally formulated by Zilber (2002, for constant families) and Pink (2005, in general). Here is a general version.

Conjecture. Let $\mathcal{G} \to T$ be a semiabelian scheme over a base T, all defined over $\mathbb{C}$. For any d, let $\mathcal{G}^{[d]}$ be the union of the semiabelian subschemes of $\mathcal{G}$ of codimension at least d. Let X be an irreducible closed subvariety of $\mathcal{G}$. Then $X^{\mathrm{na}} \cap \mathcal{G}^{[\dim X+1]}$ is contained in a finite union of semiabelian subschemes of $\mathcal{G}$ of positive codimension.

A number of people (Bombieri, Habegger, Masser, Maurin, Ratazzi, Remond, Viada, Zannier,...) have made progress on this conjecture in the last few years, especially for constant families.

Very Unlikely Intersections in $\mathbb{G}_m^r$

For the constant group scheme

$$\mathcal{G} = \mathbb{G}_m^r$$

over a variety X, Habegger's upper bound can be combined with:

Theorem. (BMZ 2008) For all B,

$$\{P \in X^{\text{na}} \cap \mathcal{G}^{[\dim X+1]} : h(P) \le B\} \quad \text{is finite.}$$

to prove finiteness:

Corollary.

$$X^{\text{na}} \cap \mathcal{G}^{[\dim X+1]} \quad \text{is finite.}$$

For non-constant families, very little is known.

Very Unlikely Intersections in a Non-Constant Family

Consider the elliptic curve

$$E : y^2 = x(x-1)(x-T)$$

and the two points

$$P = \left(2, \sqrt{4-2T}\,\right), \qquad Q = \left(3, \sqrt{18-6T}\,\right).$$

(We may view E as an elliptic curve over the function field $\mathbb{Q}(T,U,V)$, where $U^2 = 4-2T$ and $V^2 = 18-6T$.) The original theorem that we discussed says that the set of t for which P_t is a torsion point is a set of bounded height, and similarly for Q_t. What happens if we require that P_t and Q_t simultaneously be torsion points?

Theorem. (Masser–Zannier, 2008)

$$\{t \in \mathbb{C} : P_t \text{ and } Q_t \text{ are both torsion points}\}$$

is a finite set.

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