Number Theoretic Properties of Difference Equations Associated to Hénon Maps

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Arithmetic Dynamics

Discrete dynamical systems defined by polynomial or rational functions

$$\mathbf{x} = (x_1, \ldots, x_N) \longmapsto \boldsymbol{\phi}(\mathbf{x}) = \bigl(\phi_1(\mathbf{x}), \ldots, \phi_N(\mathbf{x})\bigr)$$

have been intensively studied over the past century (and more). The usual focus is dynamics in $\mathbb{R}^N$ or $\mathbb{C}^N$.

In the past 20+ years, number theorists and algebraic geometers have started studying the dynamics of such maps applied to points in $\mathbb{Z}^N$ or $\mathbb{Q}^N$, or even p-adic points $\mathbb{Q}_p^N$. This study goes by the name *Arithmetic Dynamics*.

Typical questions include:

- How many periodic points can there be in $\mathbb{Q}^N$?
- In an infinite orbit, how many points can be in $\mathbb{Z}^N$?

What I'll do today is discuss the arithmetic dynamics of difference equations and Hénon-like maps.

Rational Difference Equations and Rational Maps

Consider a rational difference equation

$$\alpha_{N+1} = R(\alpha_1, \ldots, \alpha_N), \quad \text{where} \quad R \in \mathbb{C}(x_1, \ldots, x_N).$$

There is an associated rational map

$$\phi_R : \mathbb{P}^N \longrightarrow \mathbb{P}^N$$

given in affine coordinates by

$$\phi_R(x_1, \ldots, x_N) = \bigl(x_2, \ldots, x_N, R(x_1, \ldots, x_N)\bigr).$$

The dynamics of the difference equation is mirrored by the dynamics of ϕ_R, since

$$\phi_R^n(\alpha_1, \ldots, \alpha_N) = (\alpha_{n+1}, \alpha_{n+2}, \ldots, \alpha_{n+N}).$$

Classical Hénon Maps

A classical **Hénon map** is an automorphism

$$\phi : \mathbb{A}^2 \longrightarrow \mathbb{A}^2$$

of the form

$$\phi(x,y) = (y, ax + b + y^2) \quad \text{with } a \neq 0.$$

The Hénon map is associated to the polynomial difference equation

$$\alpha_{N+1} = a\alpha_{N-1} + b + \alpha_N^2.$$

The Hénon map ϕ is an automorphism,

$$\phi^{-1}(x,y) = \left(a^{-1}(y - b - x^2), x\right).$$

The extension of ϕ to $\mathbb{P}^2$ is not a morphism, since

$$\Phi\left([X,Y,Z]\right) = [YZ, aXZ + bZ^2 + Y^2, Z^2]$$

is not defined at $[1,0,0]$.

Regular Affine Automorphisms

The Hénon map is an example of a **regular affine automorphism**, because the point $[1,0,0]$ where Φ is not defined is disjoint from the point $[0,1,0]$ where Φ^{-1} is not defined.

In general, let ϕ be a polynomial map

$$\phi = (\phi_1, \ldots, \phi_N) : \mathbb{A}^N \longrightarrow \mathbb{A}^N$$

that has a polynomial inverse, and let

$$\Phi : \mathbb{P}^N \longrightarrow \mathbb{P}^N \quad \text{and} \quad \Phi^{-1} : \mathbb{P}^N \longrightarrow \mathbb{P}^N$$

be the extensions of ϕ and ϕ^{-1} to $\mathbb{P}^N$. Then ϕ is called a **regular affine automorphism** if:

At every point P of $\mathbb{P}^N(\mathbb{C})$, at least one of $\Phi(P)$ and $\Phi^{-1}(P)$ is defined.

The real and complex dynamics of regular affine automorphisms have been extensively studied.

Periodic Points

Let

$$\phi(x, y) = (y, ax + b + y^2)$$

be a Hénon map with $a, b \in \mathbb{Q}$. Then we might ask how many of the periodic points of ϕ are in $\mathbb{Q}$.

Theorem A. A Hénon map with $a, b \in \mathbb{Q}$ has only finitely many periodic points with $x, y \in \mathbb{Q}$.

More generally, for any number field $K/\mathbb{Q}$:

Theorem B. A regular affine automorphism $\phi : \mathbb{A}^N \to \mathbb{A}^N$ defined by rational functions with coefficients in K has only finitely many periodic points in $\mathbb{A}^N(K)$.

Theorem B is due to Denis (1995) and Marcello (2000). When $\Phi : \mathbb{P}^N \to \mathbb{P}^N$ is a morphism, Theorem B was proven by Northcott in 1950.

Height Functions

Let $K/\mathbb{Q}$ be a number field. The **height** of a point $P \in \mathbb{P}^N(K)$ is

$$h(P) = \sum_{v \in M_K} \log \max_{0 \le i \le N} \|x_i(P)\|_v,$$

where the absolute values v on K are appropriately normalized.

Intuition The height $h(P)$ satisfies

$$h(P) \asymp \text{number of bits to store } P \text{ on a computer.}$$

Example For $P \in \mathbb{P}^N(\mathbb{Q})$, write

$$P = [\alpha_0, \ldots, \alpha_N] \text{ with } \alpha_i \in \mathbb{Z} \text{ and } \gcd(\alpha_0, \ldots, \alpha_N) = 1.$$

Then

$$h(P) = \log \max\{|\alpha_0|, \ldots, |\alpha_N|\}.$$

Basic Properties of Height Functions

Height functions are a fundamental tool in arithmetic geometry and arithmetic dynamics. Two important properties:

Finiteness. For all A and B, the set

$$\left\{P \in \mathbb{P}^N(\bar{\mathbb{Q}}) : h(P) \le A \text{ and } [\mathbb{Q}(P) : \mathbb{Q}] \le B\right\}$$

is finite.

Functoriality. Let

$$\phi : \mathbb{P}^N \longrightarrow \mathbb{P}^N$$

be a *morphism* defined over $\bar{\mathbb{Q}}$. Then

$$h\big(\phi(P)\big) = (\deg \phi) h(P) + O(1) \quad \text{for all } P \in \mathbb{P}^N(\bar{\mathbb{Q}}).$$

Functoriality for Rational Maps

If $\phi : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ is a rational map of degree d, we always have an upper bound

$$h\bigl(\phi(P)\bigr) \le dh(P) + O(1).$$

The proof is elementary, using the triangle inequality and a lot of algebra. But in general there is no corresponding lower bound.

Theorem. (Kawaguchi, Lee) Let $\phi : \mathbb{A}^N \to \mathbb{A}^N$ be a regular affine automorphism with $d = \deg(\phi)$ and $e = \deg(\phi^{-1})$. Then

$$\frac{1}{d}h\bigl(\phi(P)\bigr) + \frac{1}{e}h\bigl(\phi^{-1}(P)\bigr) \ge \left(1 + \frac{1}{de}\right) h(P) - C(\phi).$$

Intuition: At least one of $\phi(P)$ and $\phi^{-1}(P)$ is arithmetically much larger than P.

Uniform Bounds for Rational Periodic Points

Let

$$\phi_{a,b}(x,y) = (y, ax + b + y^2)$$

be a Hénon map with $a, b \in \mathbb{Q}$. We know that

$$\mathrm{Per}(\phi_{a,b}, \mathbb{Q}) = \{\text{periodic points of } \phi_{a,b} \text{ in } \mathbb{Q}^2\}$$

is finite. How large can it be?

Example: The map $\phi_{1,-2}$ has the rational 4-cycle

$$(0,0) \longrightarrow (0,-2) \longrightarrow (-2,2) \longrightarrow (2,0) \longrightarrow (0,0).$$

Conjecture. There is an absolute constant C such that

$$\#\,\mathrm{Per}(\phi_{a,b}, \mathbb{Q}) \le C \quad \text{for all } a \in \mathbb{Q}^*,\ b \in \mathbb{Q}.$$

This is a Hénon analogue of a uniformity conjecture that Patrick Morton and I made for morphisms of $\mathbb{P}^N$ in 1994.

Parting Questions

- Which difference equations given by rational functions with $\mathbb{Q}$-coefficients have only finitely many discrete periodic cycles whose coordinates are in $\mathbb{Q}$?
- In an algebraic family of difference equations defined over $\mathbb{Q}$, is there a uniform bound for the number of discrete periodic cycles having coordinates in $\mathbb{Q}$?

I want to thank you for your attention and to thank the organizers for inviting me to speak.

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