Algebra, Geometry, and Dynamics of Pseudo-Real Maps

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PGL_2 Equivalence of Rational Maps

The complex dynamics of a rational map

$$\phi(z) \in \mathbb{C}(z), \qquad \phi : \mathbb{P}^1(\mathbb{C}) \longrightarrow \mathbb{P}^1(\mathbb{C}),$$

is unchanged under $\mathrm{PGL}_2(\mathbb{C})$-conjugation,

$$\phi^f(z) = f^{-1} \circ \phi \circ f(z) \quad \text{for } \phi \in \mathrm{PGL}_2(\mathbb{C}).$$

The map ϕ is **defined over $\mathbb{R}$** if $\phi(z) \in \mathbb{R}(z)$, or more generally if

$$\phi^f(z) \in \mathbb{R}(z) \quad \text{for some } f \in \mathrm{PGL}_2(\mathbb{C}).$$

However, it may happen that the complex conjugate $\overline{\phi}$ of ϕ is $\mathrm{PGL}_2(\mathbb{C})$-equivalent to ϕ, even if ϕ is not defined over $\mathbb{R}$.

Quasi-Real Maps

A rational map $\phi(z) \in \mathbb{C}(z)$ is **quasi-real** if

- $\overline{\phi}(z) = \phi^f(z)$ for some $f \in \mathrm{PGL}_2(\mathbb{C})$, and
- $\phi^f(z) \notin \mathbb{R}(z)$ for all $f \in \mathrm{PGL}_2(\mathbb{C})$.

Example The map

$$\phi(z) = i\frac{z^3+1}{z^3-1}$$

is quasi-real.

Theorem. If $\phi(z) \in \mathbb{C}(z)$ is quasi-real, then:
(a) ϕ has odd degree.
(b) ϕ is not a polynomial.

This result is a special case of an old theorem on field of moduli versus fields of definition.

Quasi-Real Maps for Quadratic Extensions

More generally, if L/K is any separable extension of degree 2, we write

$$\alpha \longrightarrow \overline{\alpha} \quad \text{for the nontrivial element of } \mathrm{Gal}(L/K).$$

Then we say that $\phi(z) \in L(z)$ is **L/K-quasi-real** if

- $\overline{\phi}(z) = \phi^f(z)$ for some $f \in \mathrm{PGL}_2(L)$, and
- $\phi^f(z) \notin K(z)$ for all $f \in \mathrm{PGL}_2(L)$.

As for $\mathbb{C}/\mathbb{R}$, an L/K-quasi-real map always has odd degree and is never a polynomial.

Normalization

For ease of exposition, we generally restrict attention to maps with

$$\mathrm{Aut}(\phi) = \{f \in \mathrm{PGL}_2 : \phi^f = \phi\} = \{z\}.$$

Theorem. Let $\phi \in L(z)$ with $\mathrm{Aut}(\phi) = \{z\}$. Then the following are equivalent:

(a) L/K-quasi-real.

(b) There is a $g \in \mathrm{PGL}_2(L)$ and a $t \in K^* \smallsetminus \mathrm{N}\, L^*$ such that the map $\psi = \phi^g$ satisfies

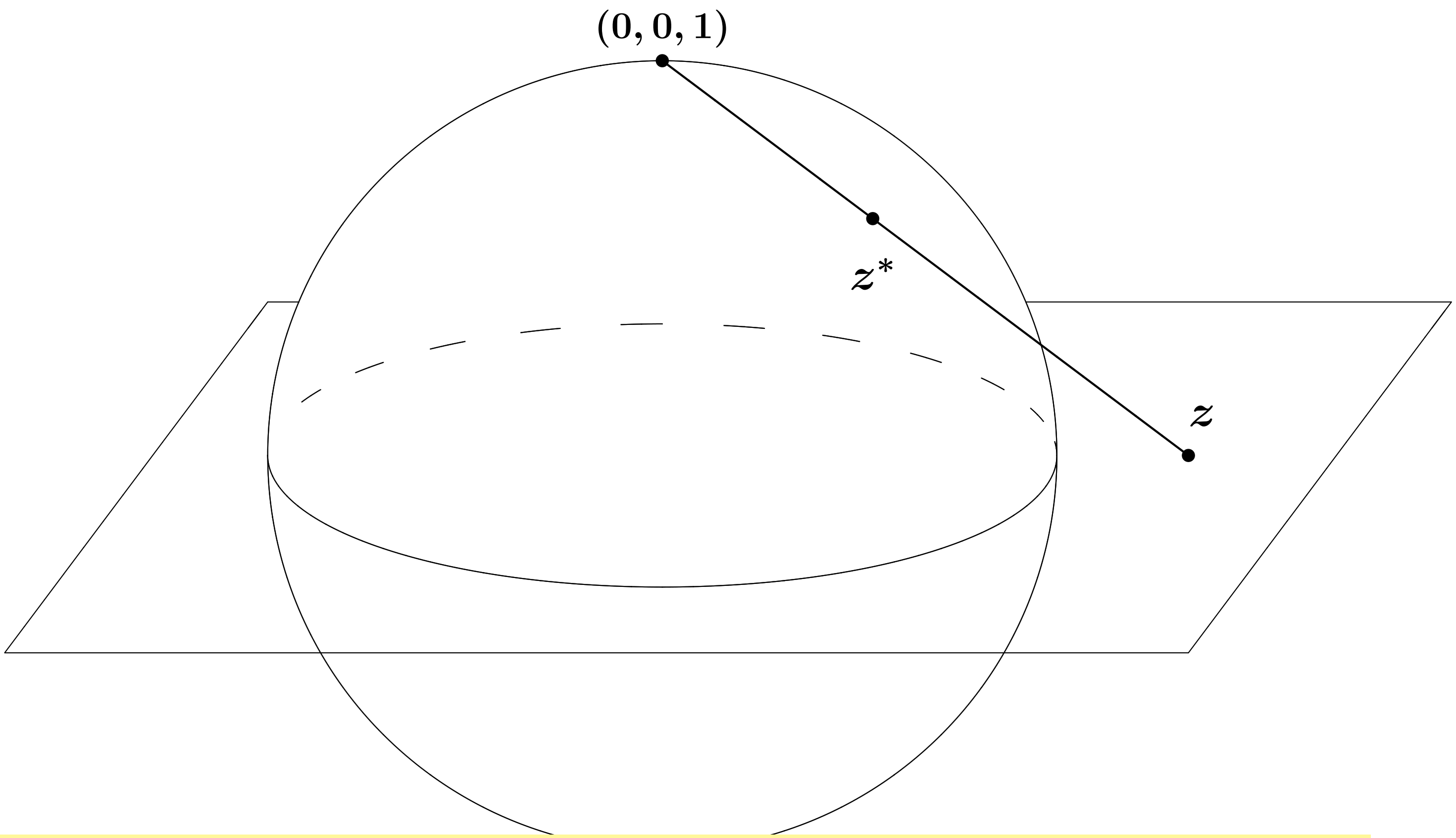
$$\overline{\psi} = \psi^f \quad \text{with } f(z) = tz^{-1}. \qquad (*)$$

The proof is a group cohomology calculation.

L/K-quasi-real maps satsifying $(*)$ are ***t*-normalized**.

Stereographic Projection

Classically, **stereographic projection** is a method of identifying $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ with the unit sphere $S^2 \subset \mathbb{C} \times \mathbb{R} = \mathbb{R}^3$. Pictorially, the identification is via



$$z = x + iy \longmapsto z^* = \left(\frac{2x + 2iy}{x^2 + y^2 + 1}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1} \right).$$

L/K Stereographic Projection

For $t \in K^* \setminus \mathrm{N}\, L^*$, we define the ***t*-sphere** to be

$$S_t(L/K) = \{(u,v) \in L \times K : v^2 = 1 + t\,\mathrm{N}\,u\}.$$

Notice $S_{-1}(\mathbb{C}/\mathbb{R})$ is the usual 2-sphere.

Then ***L*/*K* stereographic projection** is the map

$$\sigma_t : S_t(L/K) \to \mathbb{P}^1(L), \quad (u,v) \mapsto \frac{tu}{v-1}.$$

Its inverse is given by

$$\sigma_t^{-1} : \mathbb{P}^1(L) \to S_t(L/K), \quad z \mapsto \left(\frac{2z}{\mathrm{N}\,z - t}, \frac{\mathrm{N}\,z + t}{\mathrm{N}\,z - t}\right).$$

Stereographic Projection and Quasi-Real Maps

It is an algebra exercise to verify that the following diagram commutes:

$$\begin{array}{ccc} S_t(L/K) & \xrightarrow{(u,v)\mapsto(-u,-v)} & S_t(L/K) \\ \sigma_t \downarrow & & \sigma_t \downarrow \\ \mathbb{P}^1(L) & \xrightarrow{z\mapsto t\bar{z}^{-1}} & \mathbb{P}^1(L) \end{array}$$

Thus stereographic projection converts the t-inversion conjugation map on $\mathbb{P}^1(L)$ to the negation map on $S_t(L/K)$.

Another calculation shows that a t-normalized L/K-quasi-real map ϕ induces a map

$$\Phi : \frac{S_t(L/K)}{(\pm 1)} \longrightarrow \frac{S_t(L/K)}{(\pm 1)}.$$

The Induced Map on $\mathbb{P}(L \times K)$

Define

$$\mathbb{P}(L \times K) = \frac{(L \times K) \setminus (0,0)}{K^*}.$$

Clearly $\mathbb{P}(L \times K) \cong \mathbb{P}^2(K)$, but the isomorphism requires choosing a K-basis for L.
We always have

$$S_t(L/K)/(\pm 1) \hookrightarrow \mathbb{P}(L \times K),$$

and in some cases such as $\mathbb{C}/\mathbb{R}$, this map is a bijection.

Theorem. The map Φ on $S_t(L/K)/(\pm 1)$ extends to an everywhere defined algebraic map

$$\Phi : \mathbb{P}(L \times K) \longrightarrow \mathbb{P}(L \times K).$$

The proof that Φ extends to a rational map is straightforward. The fact that it is a morphism requires a more elaborate argument.

Example

Let $\mathrm{char}(K) \neq 2, 3$, let

$$t \in K^* \smallsetminus \mathrm{N}\, L^*, \quad c \in L \smallsetminus K, \quad \mathrm{N}\, c = 1, \quad c^2 \neq 1,$$

and consider the map

$$\phi(z) = \frac{c}{t}\left(\frac{z-t}{z-1}\right)^3 \in L(z).$$

Then ϕ is a t-normalized L/K quasi-real map.

Writing $-c = b/\bar{b}$ for some $b \in L$, the induced map

$$\Phi = \bigl[\Phi_1(u,v), \Phi_2(u,v)\bigr] : \mathbb{P}(L \times K) \longrightarrow \mathbb{P}(L \times K)$$

is given by

$$\begin{aligned}
\Phi_1(u,v) &= 2b^2t^7(u + t\bar{u} - 2v)^3, \\
\Phi_2(u,v) &= 2b\bar{b}t^6\bigl((3\bar{u}u^2 - 3u\bar{u}v + 3u\bar{u}^2)t^4 + (-u^3 + 3u^2v - 9u^2\bar{u} - 6uv^2 \\
&\qquad + 9u\bar{u}v - 9u\bar{u}^2 + 4v^3 - 6\bar{u}v^2 + 3\bar{u}^2v - \bar{u}^3)t^3 \\
&\qquad + (3u^2v + 3u^2\bar{u} + 9u\bar{u}v + 3u\bar{u}^2 + 3\bar{u}^2v)t^2 \\
&\qquad + (-6uv^2 - 3u\bar{u}v - 6\bar{u}v^2)t + 4v^3\bigr).
\end{aligned}$$

Wrap-Up

- A t-normalized L/K-quasi-real map is a rational function $\phi(z) \in L(z)$ satisfying
$$\overline{\phi}(z) = \frac{t}{\phi(tz^{-1})}.$$
- It induces an everywhere defined algebraic self-map
$$\mathbb{P}(L \times K) \cong \mathbb{P}^2(K). \qquad (**)$$
- **Question**: For which t (if any) can we choose the isomorphism $(**)$ so that ϕ extends to a morphism of $\mathbb{P}^2(L)$? of $\mathbb{P}^2(\bar{K})$?
- Further work-in-progress joint with Mike Zieve.

I want to thank you for your attention and to thank the organizers for inviting me to speak.

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