Dynamics and Canonical Heights on K3 Surfaces with Noncommuting Involutions

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Introductory Remarks

Our understanding of the distribution of rational points on algebraic surfaces is far from complete:

Type of Surface	Rational Points
rational and ruled	ubiquitous
abelian surface	finitely generated group
general type	not Zariski dense (conjecturally)
elliptic surfaces	various types of behavior
K3 and Enriques	???

The analysis of rational points on abelian surfaces relies on the group law. Similarly, elliptic surfaces have their fiber-by-fiber group laws, and rational surfaces have very large automorphism groups ($\mathrm{Aut}(X) = \mathrm{PGL}_3$).

In each case, geometric maps allow us to propogate rational points. It is thus natural to look at classes of K3 surfaces admitting such maps.

K3 Surfaces with Noncommuting Involutions

A Class of K3 Surfaces

Let S be a K3 surface given by the intersection of a $(2,2)$-form and a $(1,1)$-form in $\mathbb{P}^2 \times \mathbb{P}^2$.

Explicitly, the variety S is defined by a pair of bihomogeneous polynomials,

$$L(\mathbf{x},\mathbf{y}) = \sum_{0\le i\le 2} \sum_{0\le j\le 2} A_{ij} x_i y_j,$$

$$Q(\mathbf{x},\mathbf{y}) = \sum_{0\le i\le j\le 2} \sum_{0\le k\le \ell\le 2} B_{ijk\ell} x_i x_j y_k y_\ell.$$

The natural projections

$$p_1, p_2 : S \longrightarrow \mathbb{P}^2, \quad p_1(\mathbf{x},\mathbf{y}) = \mathbf{x}, \quad p_2(\mathbf{x},\mathbf{y}) = \mathbf{y},$$

have degree two, since if we fix one variable, the other is the intersection of a line and a conic in $\mathbb{P}^2$.

Noncommuting Involutions

Any double cover of varieties $p : V \to W$ induces a (rational) involution $\sigma : V \to W$ that exchanges the sheets. Thus σ is defined (generically) by

$$p^{-1}(p(Q)) = \{Q, \sigma(Q)\}.$$

We thus obtain two involutions

$$\sigma_1, \sigma_2 : S \longrightarrow S$$

corresponding to $p_1, p_2 : S \to \mathbb{P}^2$. These involutions do not commute, and their composition

$$\sigma_2 \circ \sigma_1 \in \mathrm{Aut}(S)$$

has infinite order. We denote the subgroup they generate by

$$\mathcal{A} = \langle \sigma_1, \sigma_2 \rangle \subset \mathrm{Aut}(S).$$

Orbits of Points

The set of surfaces S (up to isomorphism) forms an 18-dimensional family, and a Zariski open subset of this family consists of nonsingular sufaces such that the projections p_1 and p_2 are flat (i.e., no fiber is a a curve). We restrict attention to these surfaces.

Given any point $P \in S$, we denote the orbit of P via the group of automorphisms $\mathcal{A}$ by

$$\mathcal{A}(P) = \{\phi(P) : \phi \in \mathcal{A}\}.$$

If S is defined over K and $P \in S(K)$, then

$$\mathcal{A}(P) \subset S(K).$$

This divides the study of $S(K)$ into:

- Given P, describe its $\mathcal{A}$-orbit $\mathcal{A}(P)$.
- Describe the $\mathcal{A}$-orbits.

I will concentrate primarily on the first question.

Some Arithmetic Results Concerning $\mathcal{A}$-Orbits . . .

Assume that S is defined over a number field K. We say that $P \in S$ is **$\mathcal{A}$-periodic** if $\mathcal{A}(P)$ is finite.

Theorem 1.

$\{P \in S(K) : P \text{ is } \mathcal{A}\text{-periodic}\}$ is a finite set.

More generally, $\{P \in S(\bar{K}) : P \text{ is } \mathcal{A}\text{-periodic}\}$ is a set of bounded height.

Theorem 2. Let $P \in S(K)$ be a point with $\#\mathcal{A}(P) = \infty$, and let $h : S(\bar{K}) \to \mathbb{R}$ be the height function associated to $\mathcal{O}_S(1,1)$. Then

$$\#\{Q \in \mathcal{A}(P) : h(Q) \leq B\} = \frac{2}{\mu} \log_\alpha B + O(1),$$

where $\mu \in \{1, 2\}$ and $\alpha = 2 + \sqrt{3}$.

. . . and a Geometric Result

Theorem 3. Let $P \in S$ be a point with infinite $\mathcal{A}$-orbit. Then

$$\mathcal{A}(P) \text{ is Zariski dense in } S.$$

Here's the plan for the rest of this talk:

- Discuss the geometry of S.
- Construct canonical heights on S.
- Sketch the proofs of Theorems 1 and 2.
- Illustrate Vojta's conjecture for S.
- Describe K3 analogues of classical conjectures.
- Briefly discuss other families of K3 surfaces.
- Additional material (as time permits).

The Geometry of S

The key to understand the geometry of S is to describe the action of the involutions on $\mathrm{Pic}(S)$.

Let

$$D_1 = S \cap (H \times \mathbb{P}^2) \quad \text{and} \quad D_2 = S \cap (\mathbb{P}^2 \times H)$$

be divisors corresponding to $\mathcal{O}_S(1,0)$ and $\mathcal{O}_S(0,1)$.

Proposition. In $\mathrm{Pic}(S)$ we have

$$\begin{aligned} \sigma_1^* D_1 &= D_1 & \sigma_1^* D_2 &= 4D_1 - D_2 \\ \sigma_2^* D_1 &= -D_1 + 4D_2 & \sigma_2^* D_2 &= D_2. \end{aligned}$$

The proof is an elementary calculation using intersection theory.

Diagonalizing the Action of $\mathcal{A}$ on $\operatorname{Pic}(S) \otimes \mathbb{R}$

We diagonalize the action of σ_1 and σ_2 on $\operatorname{Pic}(S) \otimes \mathbb{R}$ by letting

$$\alpha = 2 + \sqrt{3}$$

and defining divisors

$$E^+ = \alpha D_1 - D_2 \quad \text{and} \quad E^- = -D_1 + \alpha D_2.$$

Proposition. In $\operatorname{Pic}(S) \otimes \mathbb{R}$ we have

$$\sigma_1^* E^+ = \alpha^{-1} E^- \qquad \sigma_1^* E^- = \alpha E^+$$
$$\sigma_2^* E^+ = \alpha E^- \qquad \sigma_2^* E^- = \alpha^{-1} E^+.$$

Amusing exercise: Use $\chi : \mathcal{A} \twoheadrightarrow \{\pm 1\}$ to give an action of $\mathcal{A}$ on $\mathbb{Z}$. There is a 1-cocycle $\ell : \mathcal{A} \to \mathbb{Z}$ such that

$$\phi^* E^{\pm} = \alpha^{\pm \ell(\phi)} E^{\pm \chi(\phi)} \quad \text{for all } \phi \in \mathcal{A}.$$

The Geometry of S (continued)

The divisors E^+ and E^- are on the boundary of the effective cone, but they are not themselves effective.

Proposition. Let

$$D = n_1 D_1 + n_2 D_2 \in \mathrm{Div}(S).$$

The following are equivalent:
(1) D is effective.
(2) D is ample.
(3) $D \cdot E^+ > 0$ and $D \cdot E^- > 0$.
(4) $n_1 > -\alpha n_2$ and $n_2 > -\alpha n_1$.

Corollary. If $\operatorname{rank} \mathrm{Pic}(S) = 2$ and $C \subset S$ is an irreducible curve, then

$$p_a(C) \geq 2.$$

Canonical Heights—Construction

A general construction, due essentially to Tate, says that if $\phi : V \to V$ is a morphism of varieties and if

$$\phi^* D = mD \quad \text{for some } D \in \operatorname{Pic}(V) \text{ and } m > 1,$$

then the limit

$$\hat{h}_{V,D}(P) = \lim_{k\to\infty} m^{-k} h_{V,D}\bigl(\phi^k(P)\bigr)$$

exists and satisfies

$$\begin{aligned} \hat{h}_{V,D}(P) &= h_{V,D}(P) + O(1) \\ \hat{h}_{V,D}\bigl(\phi(P)\bigr) &= m\hat{h}_{V,D}(P). \end{aligned}$$

We apply Tate's construction to

$$(\sigma_2\sigma_1)^* E^+ = \alpha^2 E^+ \qquad \text{and} \qquad (\sigma_1\sigma_2)^* E^- = \alpha^2 E^-$$

to create two canonical heights

$$\hat{h}^+ \qquad \text{and} \qquad \hat{h}^-.$$

Canonical Heights—Properties

Theorem. There are unique functions

$$\hat{h}^{+}, \hat{h}^{-} : S(\bar{K}) \longrightarrow \mathbb{R}$$

satisfying

$$\hat{h}^{\pm}(P) = h_{E^{\pm}}(P) + O(1),$$

$$\hat{h}^{\pm}(\sigma_1 P) = \alpha^{\mp 1}\hat{h}^{\mp}(P),$$

$$\hat{h}^{\pm}(\sigma_2 P) = \alpha^{\pm 1}\hat{h}^{\mp}(P).$$

Further, these canonical heights satisfy:

$\hat{h} := \hat{h}^{+} + \hat{h}^{-}$ is a Weil height for an ample divisor.

$$\hat{h}^{\pm}(P) \geq 0 \quad \text{for all } P \in S(\bar{K}).$$

$$\hat{h}^{+}(P) = 0 \Longleftrightarrow \hat{h}^{-}(P) = 0 \Longleftrightarrow \mathcal{A}(P) \text{ is finite.}$$

Canonical Heights and Periodic Points

The last chain of equivalences

$$\hat{h}^+(P) = 0 \iff \hat{h}^-(P) = 0 \iff \mathcal{A}(P) \text{ is finite} \quad (*)$$

probably looks familiar. It's an analogue of the classical result for abelian varieties:

$$\hat{h}_{A,D}(P) = 0 \iff P \in A_{\text{tors}} \qquad (D \text{ } \mathbf{ample})$$

However, $(*)$ is not immediate, because E^+ and E^- are **not** ample. So for example, the set

$$\left\{P \in S(K) : \hat{h}^+(P) < B\right\}$$

may be infinite.

The sum $\hat{h} := \hat{h}^+ + \hat{h}^-$ is relative to an ample divisor, so to prove $(*)$, it suffices to prove the first equivalence

$$\hat{h}^+(P) = 0 \iff \hat{h}^-(P) = 0.$$

Proof that $\hat{h}^+(P) = 0 \iff \hat{h}^-(P) = 0$

Suppose that $\hat{h}^+(P) = 0$, and let $\tau = \sigma_2 \circ \sigma_1$. Then

$$\begin{aligned} \hat{h}(\tau^n P) &:= \hat{h}^+(\tau^n P) + \hat{h}^-(\tau^n P) \\ &= \alpha^{2n}\hat{h}^+(P) + \alpha^{-2n}\hat{h}^-(P) \\ &= \alpha^{-2n}\hat{h}^-(P) \xrightarrow[n\to\infty]{} 0 \end{aligned}$$

Hence

$$\{\tau^n P : n \geq 0\}$$

is a set of bounded height relative to an **ample** divisor, so it is a finite set.

Then

$$\hat{h}^-(P) = \underbrace{\alpha^{2n}}_{\substack{\alpha^{2n}\to\infty \\ \text{as } n\to\infty}} \cdot \underbrace{\hat{h}^-(\tau^n P)}_{\substack{\text{finitely many} \\ \text{values for } n\geq 0}} .$$

Hence $\hat{h}^-(P) = 0$. $\square$

Proof of Finiteness of $\mathcal{A}$-periodic Points

Theorem 1.

$$\bigl\{P \in S(K) : P \text{ is } \mathcal{A}\text{-periodic}\bigr\} \text{ is a finite set.}$$

More generally, $\bigl\{P \in S(\bar{K}) : P \text{ is } \mathcal{A}\text{-periodic}\bigr\}$ is a set of bounded height.

Proof.

$$\begin{aligned}
\#\mathcal{A}(P) \text{ finite} &\iff \hat{h}^{+}(P) = \hat{h}^{-}(P) = 0 \\
&\iff \hat{h}(P) = 0 \\
&\iff h_{D_1+D_2}(P) = O(1).
\end{aligned}$$

Hence

$$\bigl\{P \in S(K) : P \text{ is } \mathcal{A}\text{-periodic}\bigr\}$$

is a set of bounded height. □

Proof of Formula Counting Points in Orbits

Theorem 2. Let $h = \frac{1}{\alpha-1} h_{D_1+D_2}$. Then

$$\#\{Q \in \mathcal{A}(P) : h(Q) \le B\} = \frac{2}{\mu} \log_\alpha B + O(1),$$

where $\mu \in \{1,2\}$ and $\alpha = 2+\sqrt{3}$.

Proof. $$h(Q) = \hat{h}^+(Q) + \hat{h}^-(Q) + O(1).$$

So writing $\mathcal{A} = \langle\tau\rangle \cup \langle\tau\rangle\sigma_1$, we have

$$h(\tau^n P) = \alpha^{2n}\hat{h}^+(P) + \alpha^{-2n}\hat{h}^-(P) + O(1),$$
$$h(\tau^n \sigma_1 P) = \alpha^{2n-1}\hat{h}^-(P) + \alpha^{-2n+1}\hat{h}^+(P) + O(1).$$

This reduces the proof to an elementary estimate for

$$\#\{n \in \mathbb{Z} : C_1\gamma^n + C_2\gamma^{-n} \le B\}$$

as a function of B. □

The Canonical Height of an $\mathcal{A}$-Orbit

More precisely, the dependence of the counting function in Theorem 2 on the point P is given by

$$\#\{Q \in \mathcal{A}(P) : h(Q) \le B\} \\ = \frac{1}{\mu} \log_\alpha \frac{B^2}{\hat{h}^+(P)\hat{h}^-(P)} + O(1),$$

where the $O(1)$ is independent of P.

The quantity

$$\hat{H}(P) = \hat{h}^+(P)\hat{h}^-(P)$$

depends only on the $\mathcal{A}$-orbit of P, so it is a natural canonical height of the orbit.

Proposition. Let $\mathcal{O}$ be any $\mathcal{A}$-orbit. Then

$$2\sqrt{\hat{H}(\mathcal{O})} \le \min_{P \in \mathcal{O}} \hat{h}(P) \le 2\alpha\sqrt{\hat{H}(\mathcal{O})}.$$

Vojta's Conjecture for K3 Surfaces

Let S be any surface with trivial canonical class and let D be an ample effective divisor. Vojta's conjecture implies that the set of integral points

$$\bigl(S \smallsetminus |D|\bigr)(R) \quad \text{is not Zariski dense.}$$

Vojta's precise statement limits the integrality of points. For our K3 surfaces in $\mathbb{P}^2 \times \mathbb{P}^2$, say over $\mathbb{Q}$, we write points $P \in S(\mathbb{Q})$ as

$$P = \bigl([x_0, x_1, x_2], [y_0, y_1, y_2]\bigr)$$
$$\text{with } x_i, y_i \in \mathbb{Z},\ \gcd(x_i) = 1,\ \gcd(y_i) = 1.$$

Vojta Conjecture. There is a $Z \subsetneq S$ so that

$$\lim_{\substack{P \in (S \smallsetminus Z)(\mathbb{Q}) \\ h(P) \to \infty}} \frac{\log \min\bigl\{|x_0|, |x_1|, |x_2|\bigr\}}{\log \max\bigl\{|x_0|, |x_1|, |x_2|\bigr\}} = 1.$$

Vojta's Conjecture: An Example

Example. Let S be the surface defined by

$$\begin{aligned} L(\mathbf{x},\mathbf{y}) &= x_0y_0 + x_1y_1 + x_2y_2, \\ Q(\mathbf{x},\mathbf{y}) &= x_0^2y_0^2 + 4x_0^2y_0y_1 - x_0^2y_1^2 + 7x_0^2y_1y_2 + 3x_0x_1y_0^2 + 3x_0x_1y_0y_1 \\ &\quad + x_0x_1y_2^2 + x_1^2y_0^2 + 2x_1^2y_1^2 + 4x_1^2y_1y_2 - x_0x_2y_1^2 \\ &\quad + 5x_0x_2y_0y_2 - 4x_1x_2y_1^2 - 4x_1x_2y_0y_2 - 2x_2^2y_0y_1 + 3x_2^2y_2^2. \end{aligned}$$

ϕ	$\phi([0,1,0],[0,0,1])$
e	$([0,1,0],[0,0,1])$
σ_2	$([1,0,0],[0,0,1])$
$\sigma_1\sigma_2$	$([1,0,0],[0,7,1])$
$\sigma_2\sigma_1\sigma_2$	$([1645,-344,2408],[0,7,1])$
$(\sigma_1\sigma_2)^2$	$([1645,-344,2408],[-13.\cdot 10^{13}, 5.6\cdot 10^{12}, 9.7\cdot 10^{12}])$
$\sigma_2(\sigma_1\sigma_2)^2$	$([2.2\cdot 10^{49}, -3.0\cdot 10^{49}, 4.6\cdot 10^{49}],[-13.\cdot 10^{13}, 5.6\cdot 10^{12}, 9.7\cdot 10^{12}])$
$(\sigma_1\sigma_2)^3$	$([2.2\cdot 10^{49}, -3.0\cdot 10^{49}, 4.6\cdot 10^{49}],[2.2\cdot 10^{186}, 1.6\cdot 10^{186}, 6.4\cdot 10^{184}])$
$\sigma_2(\sigma_1\sigma_2)^3$	$([-7.9\cdot 10^{695}, 1.0\cdot 10^{696}, 1.5\cdot 10^{695}],[2.2\cdot 10^{186}, 1.6\cdot 10^{186}, 6.4\cdot 10^{184}])$

It is striking how the three $\mathbf{x}$-coordinates have the same order of magnitude, and similarly for the $\mathbf{y}$-coordinates (as predicted by Vojta's conjecture).

Vojta's Conjecture for Orbits

The proof of Vojta's conjecture, even for K3 surfaces $S \subset \mathbb{P}^2 \times \mathbb{P}^2$, is currently out of reach. Might it be easier if we restrict attention to points lying in an orbit?

Speculation. Siegel, Vojta, and Faltings developed geometric and Diophantine approximation methods to study integral and rational points on curves and, more generally, on subvarieties of abelian varieties. These techniques rely on the group law. Might it be possible to use the group of automorphisms $\mathcal{A}$ in place of the group law on an abelian variety to prove:

Vojta Conjecture for Orbits. Fix $Q \in S(\mathbb{Q})$ with $\#\mathcal{A}(Q) = \infty$. Then

$$\lim_{P \in \mathcal{A}(Q)} \frac{\log \min\bigl\{|x_0|, |x_1|, |x_2|\bigr\}}{\log \max\bigl\{|x_0|, |x_1|, |x_2|\bigr\}} = 1.$$

K3 Analogues of Some Classical Conjectures

In the following, all K3 surfaces are $S \subset \mathbb{P}^2 \times \mathbb{P}^2$ with $\mathcal{A} = \langle \sigma_1, \sigma_2 \rangle \subset \operatorname{Aut}(S)$. We define

$$S[\mathcal{A}] = \{P \in S : \mathcal{A}(P) \text{ is finite}\}.$$

K3 Uniform Boundedness Conjecture. There is a constant $c = c(K)$ such that for all K3 surfaces S/K,

$$\#S[\mathcal{A}](K) \le c.$$

K3 Manin–Mumford Conjecture. Let $C \subset S$ be a curve such that $\phi(C) \ne C$ for all $\phi \in \mathcal{A}$. Then

$$C \cap S[\mathcal{A}] \text{ is finite.}$$

(Weak) K3 Lehmer Conjecture. Fix S/K. There are constants $c = c(S/K) > 0$ and $\delta = \delta(S/K)$ so that

$$\hat{h}(P) \ge \frac{c}{[L:K]^{\delta}} \quad \text{for all } L/K \text{ and } P \in S(L) \setminus S[\mathcal{A}].$$

K3 Analogues of Classical Conjectures (continued)

K3 Lang Height Conjecture. There is a constant $c = c(K)$ such that for all K3 surfaces S/K,

$$\hat{h}(P) \ge ch(S) \qquad \text{for all } P \in S(K) \setminus S[\mathcal{A}].$$

(Here $h(S)$ is the height of S as a point in the moduli space of K3 surfaces.)

K3 Serre Image-of-Galois Conjecture. For any subgroup $\mathcal{B} \subset \mathcal{A}$, let

$$S_{\mathcal{B}} := \{P \in S(\bar{K}) : \mathcal{B} \text{ is the stabilizer of } P \text{ in } \mathcal{A}\},$$

and define

$$\rho_{\mathcal{B}} : \operatorname{Gal}\bigl(K(S_{\mathcal{B}})/K\bigr) \longrightarrow \operatorname{SymmGp}(S_{\mathcal{B}}).$$

There is a constant $c = c(S/K)$ so that for all subgroups $\mathcal{B} \subset \mathcal{A}$ of *finite index*,

$$\bigl(\operatorname{SymmGp}(S_{\mathcal{B}}) : \operatorname{Image}(\rho_{\mathcal{B}})\bigr) < c$$

A Family of K3 Surfaces with Three Involutions

Let

$$S \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

be a (smooth) hypersurface defined by a $(2,2,2)$-form.

Then each of the projections $(1 \le i < j \le 3)$

$$p_{ij} : S \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1, \qquad (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = (\mathbf{x}_i, \mathbf{x}_j)$$

is a double cover, so induces an involution

$$\sigma_{ij} : S \longrightarrow S.$$

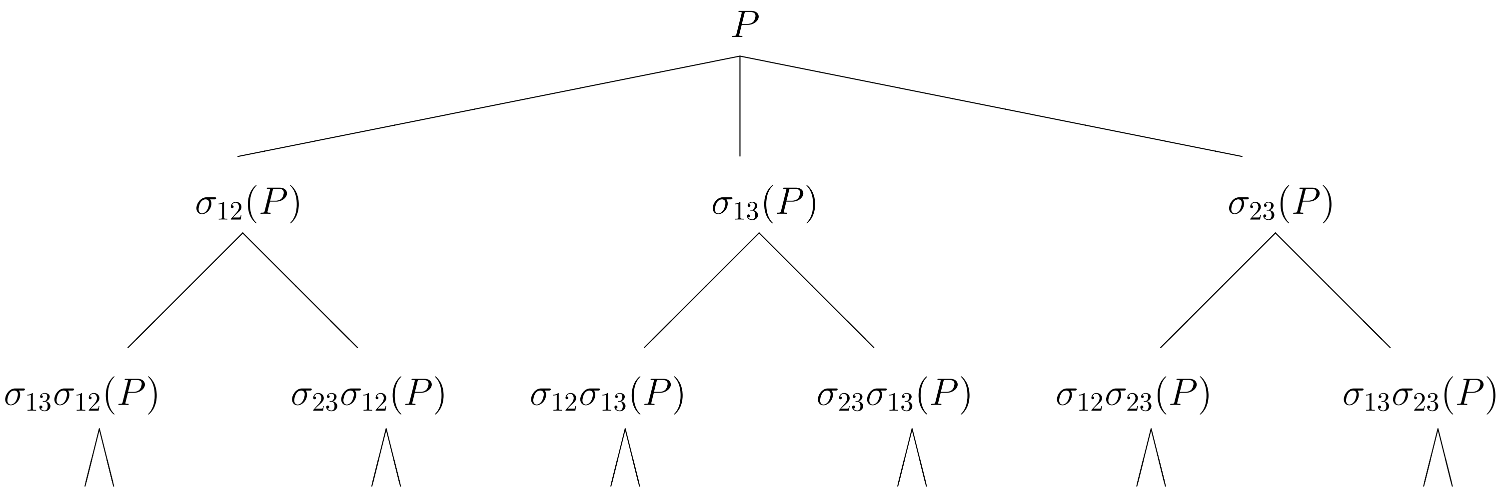
The group of automorphisms

$$\mathcal{A} = \langle \sigma_{12}, \sigma_{13}, \sigma_{23} \rangle$$

has a subgroup of index two that is isomorphic to $\mathbb{Z} \star \mathbb{Z}$, the free product of two copies of $\mathbb{Z}$.

A Family of K3 Surfaces with Three Involutions

The orbit $\mathcal{A}(P)$ of a point $P \in S$ is (generally) a tree of valency 3.



The orbit structure of points on these surfaces is reminiscent of the orbit structure associated to Markoff numbers, which are (integer) solutions to the Markoff equation

$$x^2 + y^2 + z^2 = 3xyz.$$

Canonical Heights and Counting Points

The arithmetic properties of this family of K3 surfaces was first studied by Arthur Baragar and Lan Wang. The Picard group of S has rank at least 3, generated by the three pullbacks

$$D_{ij} = p_{ij}^*(\text{hyperplane}).$$

Diagonalizing the action of specific $\phi \in \mathcal{A}$ on the subgroup of $\text{Pic}(S)$ generated by the D_{ij} yields heights that are canonical relative to $\langle\phi\rangle$.

Theorem. (Baragar) There is an open set $U \subset S$ so that if $P \in S(K)$ satisfies $\mathcal{A}(P) \subset U$, then

$$\#\{Q \in \mathcal{A}(P) : h(Q) \le B\} \gg\ll B.$$

(Here h is a height with respect to $D_{12} + D_{13} + D_{23}$.)

Canonical Vector Heights

Baragar has defined a notion of **canonical vector height**. This is a divisor class-valued function

$$\hat{\boldsymbol{h}} : S(K) \longrightarrow \mathrm{Pic}(S/K) \otimes \mathbb{R}$$

such that for all

$$P \in S(K), \quad D \in \mathrm{Pic}(S/K) \otimes \mathbb{R}, \quad \sigma \in \mathrm{Aut}(S),$$

we have

$$\hat{\boldsymbol{h}}(P) \cdot D = h_D(P) + O(1),$$
$$\hat{\boldsymbol{h}}(\sigma P) = \sigma_* \hat{\boldsymbol{h}}(P).$$

Baragar proves that $\hat{\boldsymbol{h}}$ exists on K3 surfaces with Picard number 2, and recently, he and van Luijk have shown that canonical vector heights do not exist on certain K3 surfaces of Picard number 3.

Proof of Theorem 3

Theorem 3. Let $P \in S$ be a point with infinite $\mathcal{A}$-orbit. Then $\mathcal{A}(P)$ is Zariski dense in S.

Proof sketch. If $\mathcal{A}(P)$ is not Zariski dense, write

$$\overline{\mathcal{A}(P)} = C_1 \cup C_2 \cup \cdots \cup C_n, \quad \text{a union of irreducible curves.}$$

Let $\tau = \sigma_2\sigma_1 \in \mathcal{A}$. For each C_i we have

$$\infty = \#\big(\mathcal{A}(P) \cap C_i\big) = \#\big(\tau\mathcal{A}(P) \cap C_i\big) = \#\big(\mathcal{A}(P) \cap \tau^{-1}C_i\big).$$

Hence $\tau^{-1}C_i$ equals some C_j, so τ permutes $C_1, \ldots, C_n$. Thus there is a $k \geq 1$ such that $\tau^k C_1 = C_1$.

$$E^{\pm} \cdot C_1 = E^{\pm} \cdot \tau^k C_1 = \big(\tau^k\big)^* E^{\pm} \cdot C_1 = \alpha^{\pm 2k} E^{\pm} \cdot C_1.$$

Therefore $E^+ \cdot C_1 = E^- \cdot C_1 = 0$.
But $E^+ + E^-$ is ample, which is a contradiction. $\square$

Computing $\hat{h}^{\pm}$ on $S \subset \mathbb{P}^2 \times \mathbb{P}^2$

If you like to do computations, see:

G. Call and J.H. Silverman, Computing the canonical height on $K3$ surfaces, *Math. Comp.* **65** (1996), 259–290.

- Criteria to check that S is nonsingular.
- Criteria to check that σ_1 and σ_2 are morphisms.
- Algorithms for the involutions σ_1 and σ_2.
- Algorithms for local height functions $\hat{\lambda}^+$ and $\hat{\lambda}^-$.

This allows the practical computation of canonical heights

$$\hat{h}^{\pm}(P) = \sum_{v \in M_K} \hat{\lambda}^{\pm}(P; v).$$

Let $P = \bigl([0,1,0],[0,0,1]\bigr)$ be on the earlier surface S.

$$\hat{h}^{+}(P) = 0.14758 \qquad \text{and} \qquad \hat{h}^{-}(P) = 0.55076.$$

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