# The Arithmetic of Dynamical Systems

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# What Is Dynamics?

Abstractly, a **(Discrete) Dynamical System** is a map

$$\phi : S \longrightarrow S$$

from a set to itself. Dynamics is the study of the behavior of the points in $S$ under iteration of the map $\phi$.

We write

$$\phi^n = \underbrace{\phi \circ \phi \circ \phi \cdots \phi}_{n \text{ iterations}}$$

for the $n^{\text{th}}$ iterate of $\phi$ and

$$\mathcal{O}_\phi(\alpha) = \{\alpha, \phi(\alpha), \phi^2(\alpha), \phi^3(\alpha), \dots\}$$

for the **(forward) orbit of** $\alpha \in S$.

A primary goal in the study of dynamics is to classify the points of $S$ according to the behavior of their orbits.

## Rational Maps

In this talk I will concentrate on rational functions

$$\phi(z) = \frac{F(z)}{G(z)} = \frac{a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0}{b_e z^e + b_{e-1} z^{e-1} + \cdots + b_1 z + b_0}$$

that define maps

$$\phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$$

on the projective line, although I will also mention maps

$$\phi : \mathbb{P}^N \longrightarrow \mathbb{P}^N$$

on higher dimensional projective spaces, and possibly maps $\phi : V \to V$ on other algebraic varieties.

The **degree** of the rational map $\phi$ is the larger of $d$ and $e$, where $a_d \neq 0$ and $b_e \neq 0$. The rational maps in this talk will be assumed to have degree at least 2.

# A Soupçon of Classical Dynamics

## Periodic and Preperiodic Points

A **Periodic Point** is a point that returns to where it started,

$$\alpha \xrightarrow{\phi} \phi(\alpha) \xrightarrow{\phi} \phi^2(\alpha) \xrightarrow{\phi} \cdots \xrightarrow{\phi} \phi^n(\alpha) = \alpha.$$

The smallest such $n$ is the **Period of $\alpha$**

Another interesting set of points consists of those with a finite orbit,

$$\#\mathcal{O}_\phi(\alpha) < \infty.$$

These are called **Preperiodic Points**.

We write

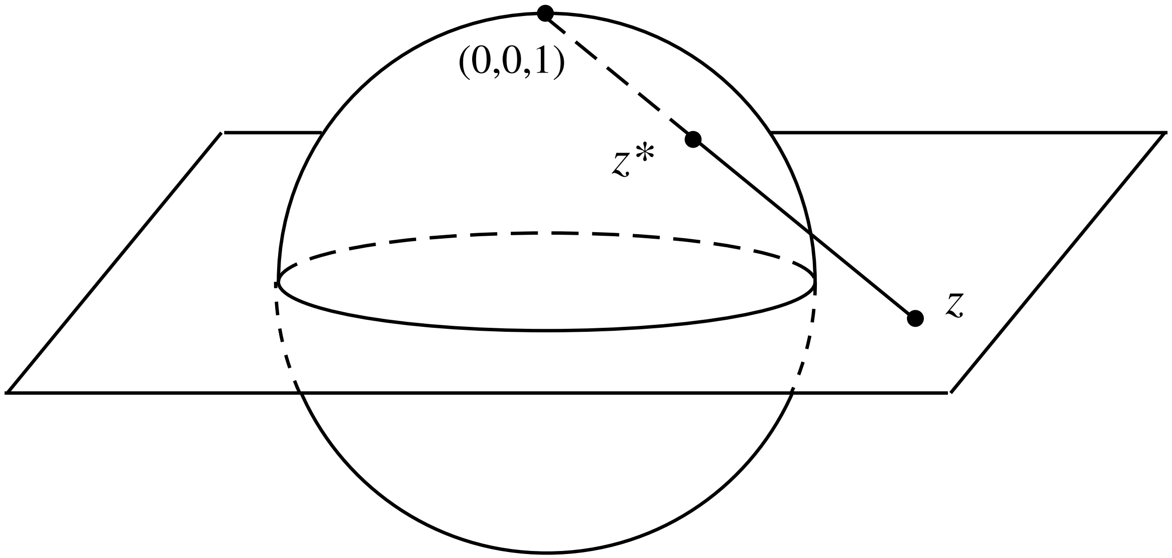
$$\mathrm{Per}(\phi) \qquad \text{and} \qquad \mathrm{PrePer}(\phi)$$

for the sets of periodic and preperiodic points of $\phi$.

Periodic and preperiodic points are defined by purely algebraic properties. Things get more interesting when we add in a topology or a metric.

## The Chordal Metric on $\mathbb{P}^1(\mathbb{C})$

A natural metric on the complex projective plane is the chordal metric, which is defined by identifying $\mathbb{P}^1(\mathbb{C})$ with the Riemann sphere.




Each point $z \in \mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \infty$ is identified with a point $z^*$ on the sphere. The chordal metric is

$$\rho(z,w) = \frac{1}{2}|z^* - w^*| = \frac{|z-w|}{\sqrt{|z|^2+1}\sqrt{|w|^2+1}}.$$

## The Fatou and Julia Sets

Some points $\alpha \in \mathbb{P}^1(\mathbb{C})$ have the property that if $\beta$ starts close to $\alpha$, then all of the iterates $\phi^n(\beta)$ stay close to the corresponding $\phi^n(\alpha)$. Informally, the **Fatou Set of $\phi$** is the set of all $\alpha \in \mathbb{P}^1(\mathbb{C})$ satisfying

$$\beta \text{ close to } \alpha \implies \phi^n(\beta) \text{ close to } \phi^n(\alpha) \text{ for all } n \geq 0.$$

The **Julia Set of $\phi$** is the complement of the Fatou set. Points in the Julia set tend to move away from one another under iteration of $\phi$. They behave **chaotically**.

More formally, the Fatou set $\text{Fatou}(\phi)$ is the largest open subset of $\mathbb{P}^1(\mathbb{C})$ on which the set of iterates $\{\phi^n\}_{n\geq 0}$ is an equicontinuous family of maps.

A major goal of complex dynamics is to describe the Fatou and Julia sets for various types of maps.

# What Is Arithmetic Dynamics?

## Arithmetic Dynamics and Arithmetic Geometry

**Arithmetic Dynamics** is the study of number theoretic properties of dynamical systems. The transposition of classical results from arithmetic geometry and the theory of Diophantine equations to the setting of discrete dynamical systems leads to a host of new and interesting questions. Although there is no precise dictionary connecting the two areas, the following associations give a flavor of the correspondence:

| **Diophantine Equations** | | **Dynamical Systems** |
|---|---|---|
| rational and integral points on varieties | ⟷ | rational and integral points in orbits |
| torsion points on abelian varieties | ⟷ | periodic and preperiodic points of rational maps |

## Two Motivating Problems in Arithmetic Dynamics

Let $\phi \in \mathbb{Q}(z)$.

**Rationality of Periodic Points**

To what extent can the periodic or preperiodic points of $\phi$ be in $\mathbb{Q}$? The Diophantine analogue is $\mathbb{Q}$-rationality of torsion points on elliptic curves (Mazur, Merel) and abelian varieties.

**Integral Points in Orbits**

To what extent can orbit $\mathcal{O}_\phi(\alpha)$ contain infinitely many integer points? The Diophantine analogue is integral points on curves (Siegel) and on higher dimensional varieties (Faltings, Vojta).

More generally, one can ask the same questions for morphisms

$$\phi : \mathbb{P}^N \longrightarrow \mathbb{P}^N$$

defined over a number field $K$.

# Rationality of Periodic Points

## Northcott's Theorem

**Theorem.** (Northcott 1949) *Let $\phi : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism defined over a number field $K$. Then*

$$\operatorname{PrePer}(\phi) \cap \mathbb{P}^N(K) \text{ is finite.}$$

*In particular, a rational function $\phi(z) \in \mathbb{Q}(z)$ of degree $d \geq 2$ has only finitely many rational periodic points.*

The proof is an easy application of the theory of heights.

Northcott's theorem is the dynamical analog of the fact that if $A/K$ is an abelian variety, then the torsion subgroup

$$A(K)_{\text{tors}} \text{ is finite.}$$

## The Uniform Boundedness Conjecture

It is natural to ask: How large can we make the set

$$\operatorname{PrePer}(\phi) \cap \mathbb{P}^N(K)? \qquad (*)$$

It is easy to make $(*)$ large if we allow $\deg(\phi)$ or $N$ or $K$ to grow. Otherwise we have the…

**Uniform Boundedness Conjecture.** (Morton–Silverman) *There is a constant $C = C(D, d, N)$ so that for all fields $K/\mathbb{Q}$ of degree $D$ and all morphisms*

$$\phi : \mathbb{P}^N \longrightarrow \mathbb{P}^N$$

*of degree $d$ defined over $K$ we have*

$$\#\operatorname{PrePer}(\phi) \cap \mathbb{P}^N(K) \le C.$$

## Consequences of the Uniform Boundedness Conjecture

The Uniform Boundedness Conjecture for

$$(D, d, N) = (1, 4, 1)$$

implies **Mazur's Theorem**: *There is an absolute constant $C$ so that for all elliptic curves $E/\mathbb{Q}$,*

$$\#E(\mathbb{Q})_{\text{tors}} \le C.$$

The implication is trivial, using the rational map $\phi$ determined by the equation $\phi\bigl(x(P)\bigr) = x\bigl([2]P\bigr)$ and the fact that $E_{\text{tors}} = \operatorname{PrePer}\bigl([2]\bigr)$.

Fakhruddin has shown the Uniform Boundedness Conjecture implies uniform boundedness for abelian varieties:

**Conjecture**. *For all fields $K/\mathbb{Q}$ of degree $D$ and all abelian varieties $A/K$ of dimension $N$,*

$$\#A(K)_{\text{tors}} \le C(D, N).$$

## Uniform Boundedness for $\phi_c(z) = z^2 + c$

Quadratic polynomials

$$\phi_c(z) = z^2 + c$$

provide the first nontrivial cases of dynamical systems and have been used as a testing ground since the genesis of the subject.

Even for this family, little is known about rationality of periodic points. Here is our meagre state of knowledge:

| $n$ | $\phi_c$ may have $\mathbb{Q}$-rational point of period $n$? |
|---|---|
| 1 | YES — 1 parameter family |
| 2 | YES — 1 parameter family |
| 3 | YES — 1 parameter family |
| 4 | NO (Morton) |
| 5 | NO (Flynn–Poonen–Schaefer) |
| $\geq 6$ | ??? |

## Dynamical Modular Curves

Consider the moduli problem of classifying pairs

$$\{(c, \alpha) : \alpha \text{ has exact period } n \text{ for } \phi_c(z) = z^2 + c\}.$$

This is analogous to the classical moduli problem of classifying pairs $(E, P)$, where $E$ is an elliptic curve and $P$ is a point of exact period $n$ on $E$.

As in the classical case, there is an affine curve $Y_1^{\text{dyn}}(n)$ that (almost) solves the dynamical moduli problem. We can summarize the table on the previous slide:

$$Y_1^{\text{dyn}}(1) \cong Y_1^{\text{dyn}}(2) \cong Y_1^{\text{dyn}}(3) \cong \mathbb{P}^1,$$
$$Y_1^{\text{dyn}}(4)(\mathbb{Q}) = Y_1^{\text{dyn}}(5)(\mathbb{Q}) = \emptyset.$$

Unfortunately, there does not seem to be a dynamical analog for the ring of Hecke operators, which is so important in studying elliptic modular curves.

# Integral Points in Orbits

# Integral Points in Orbits

A famous theorem of Siegel says that a polynomial equation

$$f(x, y) = 0$$

has only finitely many solutions $(x, y) \in \mathbb{Z}$ if the associated Riemann surface has genus at least 1.

A natural dynamical analog: Given $\phi(z) \in \mathbb{Q}(z)$ and $\alpha \in \mathbb{Q}$, can

$$\#\bigl(\mathcal{O}_\phi(\alpha) \cap \mathbb{Z}\bigr) = \infty?$$

The answer is clearly "Yes". For example, if $\phi(z) \in \mathbb{Z}[z]$ and $\alpha \in \mathbb{Z}$, then $\mathcal{O}_\phi(\alpha) \subset \mathbb{Z}$.

Similarly, if

$$\phi(z) = 1/z^d, \text{ then } \phi^2(z) = z^{d^2},$$

so $\mathcal{O}_\phi(\alpha)$ may contain infinitely many integer points.

## Integral Points in Orbits

More generally, there may be infinitely many integer points in $\mathcal{O}_\phi(\alpha)$ if any iterate $\phi^n(z)$ is a polynomial. However, it turns out...

**Proposition.** *Let* $\phi(z) \in \mathbb{C}(z)$. *If* $\phi^n(z) \in \mathbb{C}[z]$ *for some* $n \geq 1$, *then already* $\phi^2(z) \in \mathbb{C}[z]$.

The proof is a nice exercise using the Riemann-Hurwitz genus formula.

Ruling out polynomial iterates leads to the correct statement:

**Theorem.** *Let* $\phi(z) \in \mathbb{Q}(z)$ *with* $\phi^2(z) \notin \mathbb{Q}[z]$ *and let* $\alpha \in \mathbb{Q}$. *Then*

$$\#\bigl(\mathcal{O}_\phi(\alpha) \cap \mathbb{Z}\bigr) \text{ is finite.}$$

## A Refined Integrality Result

Let $\phi(z) \in \mathbb{Q}(z)$ and $\alpha \in \mathbb{Q}$, and write

$$\phi^n(\alpha) = \frac{A_n}{B_n} \in \mathbb{Q}$$

as a fraction in lowest terms.

**Theorem.** *Assume that*

$$\phi^2(z) \notin \mathbb{Q}[z] \quad \textit{and that} \quad 1/\phi^2(1/z) \notin \mathbb{Q}[z].$$

*Then*

$$\lim_{n\to\infty} \frac{\log|A_n|}{\log|B_n|} = 1.$$

The proof is an adaptation of Siegel's argument, with extra complications due to the fact that the map $\phi$ is ramified. Ultimately it depends on a result from Diophantine approximation such as Roth's theorem.

# Canonical Height Functions

## Height Functions

**Height Functions** are a fundamental tool in the study of Diophantine equations. The height of a rational number written in lowest terms is

$$h\left(\frac{a}{b}\right) = \log\max\bigl\{|a|, |b|\bigr\}.$$

The height measures the *arithmetic complexity* of $\frac{a}{b}$.

More generally, a point $P \in \mathbb{P}^N(\mathbb{Q})$ can be written as

$$P = [a_0, a_1, \ldots, a_N] \quad \text{with } a_i \in \mathbb{Z} \text{ and } \gcd(a_i) = 1.$$

Then the **height of $P$** is

$$h(P) = \log\max\bigl\{|a_0|, |a_1|, \ldots, |a_N|\bigr\}.$$

Even more generally, there is a natural way to extend the height funtion to all algebraic numbers,

$$h : \mathbb{P}^N(\bar{\mathbb{Q}}) \longrightarrow [0, \infty).$$

## Elementary Properties of Height Functions

Height functions provide a tool for translating geometric facts into arithmetic facts. Two key properties:

- For any number field $K/\mathbb{Q}$ and any $B$, the set

$$\{P \in \mathbb{P}^N(K) : h(P) \le B\} \quad \text{is finite.}$$

- For any morphism $\phi : \mathbb{P}^N \longrightarrow \mathbb{P}^N$ of degree $d \ge 2$,

$$h\bigl(\phi(P)\bigr) - d \cdot h(P) \quad \text{is bounded.} \qquad (*)$$

It is reasonable that $(*)$ is bounded, since $\phi$ is defined by polynomials of degree $d$, and clearly $h(\alpha^d) = d \cdot h(\alpha)$. But the lower bound $h\bigl(\phi(P)\bigr) \ge d \cdot h(P) + O(1)$ is non-trivial and requires some version of the Nullstellensatz.

Notice how property $(*)$ translates geometric information ($\deg \phi = d$) into arithmetic information ($\phi(P)$ is approximately $d$-times as arithmetically complex as $P$).

## Canonical Height Functions

It would be nice to replace property $(*)$ with an equality.

**Theorem.** (Tate) *The limit*

$$\hat{h}_\phi(P) = \lim_{n\to\infty} \frac{1}{d^n} h(\phi^n(P))$$

*exists and has the following properties:*

(a) $\hat{h}_\phi(P) = h(P) + O(1)$ *for all* $P \in \mathbb{P}^N(\bar{\mathbb{Q}})$.

(b) $\hat{h}_\phi(\phi(P)) = d \cdot h(P)$ *for all* $P \in \mathbb{P}^N(\bar{\mathbb{Q}})$.

Note that (a) says $\hat{h}_\phi$ measures arithmetic complexity and that (b) says $\hat{h}_\phi$ transforms canonically.

**Proposition** *Let* $P \in \mathbb{P}^N(\bar{\mathbb{Q}})$. *Then* $\hat{h}_\phi(P) \ge 0$, *and*

$$\hat{h}_\phi(P) = 0 \quad\Longleftrightarrow\quad P \in \operatorname{PrePer}(\phi).$$

# Canonical Heights and an Arithmetic Distance Function

Joint work with
Shu Kawaguchi

## An Arithmetic Distance Function

The canonical height function

$$\hat{h}_\phi : \mathbb{P}^N(\bar{\mathbb{Q}}) \longrightarrow [0,\infty)$$

encodes a large amount of arithmetic-dynamical information about the map $\phi$. It is natural to measure the arithmetic-dynamical "distance" between two maps $\phi$ and $\psi$ by

$$\delta(\phi,\psi) = \sup_{P\in\mathbb{P}^N(\bar{\mathbb{Q}})} \left|\hat{h}_\phi(P) - \hat{h}_\psi(P)\right|.$$

In particular, this raises the natural

**Question**. When does $\delta(\phi,\psi)=0$, i.e., for what maps $\phi$ and $\psi$ is it true that

$$\hat{h}_\phi(P) = \hat{h}_\psi(P) \quad \text{for all } P \in \mathbb{P}^N(\bar{\mathbb{Q}})?$$

## Maps with Identical Canonical Heights

**Theorem.** (SK–JS) *Let $\phi, \psi : \mathbb{P}^N \to \mathbb{P}^N$ be morphisms of degree $\ge 2$ defined over $\bar{\mathbb{Q}}$ and fix an embedding $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Suppose that*

$$\hat{h}_\phi = \hat{h}_\psi.$$

*Then*

$$\operatorname{Julia}(\phi) = \operatorname{Julia}(\psi) \quad \textit{in } \mathbb{P}^N(\mathbb{C}).$$

*Proof Sketch.* Write the canonical height $\hat{h}_\phi$ as a sum

$$\hat{h}_\phi(P) = \sum\nolimits_v G_{v,\phi}(P)$$

of $v$-adic Green functions (local heights). Use weak approximation to prove that $G_{v,\phi} = G_{v,\psi}$ for every absolute value $v$. Finally use the fact that $\operatorname{Julia}(\phi)$ is the support of the current attached to $G_{\infty,\phi}$.

## Polynomials with Identical Canonical Heights

In some cases, we can say considerably more.

**Corollary.** *Let $\phi, \psi \in \bar{\mathbb{Q}}[z]$ be polynomials satisfying*

$$\hat{h}_\phi = \hat{h}_\psi.$$

*Then after a change of variables (conjugation by $Az + B$), one of the following is true:*

(a) $\phi(z) = az^d$ *and* $\psi(z) = bz^e$ *are monomials.*

(b) $\phi(z) = \pm T_d(z)$ *and* $\psi(z) = \pm T_e(z)$ *are Chebyshev polynomials.*

(c) *There is a* $\lambda(z) \in \bar{\mathbb{Q}}[z]$ *so that* $\phi(z) = \lambda^n(z)$ *and* $\psi(z) = \lambda^m(z)$.

## Power Maps and Canonical Heights

Another case in which we can say more is the $d^{\text{th}}$ power map

$$[x_0, \ldots, x_N] \longmapsto [x_0^d, \ldots, x_N^d].$$

The ordinary height $h$ is already canonical for this map.

**Theorem.** (SK–JS) *A map* $\phi : \mathbb{P}^N \to \mathbb{P}^N$ *satisfies*

$$\hat{h}_\phi = h$$

*if and only if there is a change of variables (conjugation by an element of* $\operatorname{Aut}(\mathbb{P}^N) = \operatorname{PGL}_{N+1}$*) so that some iterate of* $\phi$ *has the form*

$$\phi^n = [\xi_0 x_0^e, \ldots, \xi_N x_N^e]$$

*with roots of unity* $\xi_0, \ldots, \xi_N$.

# Moduli Spaces for Dynamical Systems

## Rational Maps and $\mathrm{PGL}_2$ Conjugation

Consider the set of rational maps of degree $d$,

$$\mathrm{Rat}_d = \{\text{degree } d \text{ rational maps } \phi : \mathbb{P}^1 \to \mathbb{P}^1\}.$$

For $\phi \in \mathrm{Rat}_d$ and $f \in \mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2$, we write

$$\phi^f = f^{-1} \circ \phi \circ f.$$

Notice that iteration of $\phi^f$ is given by

$$(\phi^f)^n = (f^{-1}\phi f) \cdots (f^{-1}\phi f) = f^{-1}\phi^n f = (\phi^n)^f,$$

so the dynamics of $\phi$ and its conjugate $\phi^f$ are identical.

This makes it natural to look at the space of rational maps modulo the conjugation action of $\mathrm{PGL}_2$. We denote this set by

$$\mathcal{M}_d = \mathrm{Rat}_d \,/\, \mathrm{PGL}_2\,.$$

## $\mathrm{Rat}_d$ and $\mathcal{M}_d$ as Algebraic Varieties

If we write $\phi \in \mathrm{Rat}_d$ as

$$\phi(z) = \frac{a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0}{b_d z^d + b_{d-1} z^{d-1} + \cdots + b_1 z + b_0} = \frac{F_{\mathbf{a}}(z)}{F_{\mathbf{b}}(z)},$$

then $\mathrm{Rat}_d$ is naturally an algebraic variety,

$$\mathrm{Rat}_d = \bigl\{ [a_0, \ldots, b_d] \in \mathbb{P}^{2d+1} : \mathrm{Res}(F_{\mathbf{a}}, F_{\mathbf{b}}) \neq 0 \bigr\}.$$

**Theorem.** (a) (Milnor) *The quotient*

$$\mathcal{M}_d(\mathbb{C}) = \mathrm{Rat}_d(\mathbb{C}) / \mathrm{PGL}_2(\mathbb{C})$$

*has a natural structure as a complex orbifold.*
(b) (JS) *The quotient*

$$\mathcal{M}_d = \mathrm{Rat}_d / \mathrm{PSL}_2$$

*has a natural structure as a scheme over* $\mathbb{Z}$*.*

## The Geometry of the Variety $\mathcal{M}_d$

The construction of $\mathcal{M}_d$ as a variety, or more generally as a scheme over $\mathbb{Z}$, uses Mumford's geometric invariant theory.

This theory provides two larger quotient spaces $\mathcal{M}_d^s$ and $\mathcal{M}_d^{ss}$ with the following useful properties:

$$\mathcal{M}_d^s(\mathbb{C}) = \frac{\mathrm{Rat}_d^s(\mathbb{C})}{\mathrm{PGL}_2(\mathbb{C})} \quad \text{and} \quad \mathcal{M}_d^{ss}(\mathbb{C}) \text{ is compact.}$$

**Proposition.** *If $d$ is even, then*

$$\mathcal{M}_d^s = \mathcal{M}_d^{ss},$$

*so in this case there is a geometric quotient space, that is a natural compactification $\mathcal{M}_d$.*

For even $d$, we write $\overline{\mathcal{M}}_d$ for $\mathcal{M}_d^s = \mathcal{M}_d^{ss}$.

## The Moduli Space of Rational Maps of Degree 2

The geometry of $\mathcal{M}_d$ is not well understood, but for $d = 2$ we have...

**Theorem.**
(a) (Milnor) $\mathcal{M}_2(\mathbb{C}) \cong \mathbb{C}^2$ *and* $\overline{\mathcal{M}}_2(\mathbb{C}) \cong \mathbb{CP}^2$.
(b) (JS) $\mathcal{M}_2 \cong \mathbb{A}^2$ *and* $\overline{\mathcal{M}}_2 \cong \mathbb{P}^2$ *as varieties over* $\mathbb{Q}$*, and in fact as schemes over* $\mathbb{Z}$.

The isomorphism $(\sigma_1, \sigma_2) : \mathcal{M}_2 \to \mathbb{A}^2$ is quite explicit, but rather complicated:

$$\sigma_1 = \frac{\begin{aligned} &a_1^3b_0 - 4a_0a_1a_2b_0 - 6a_2^2b_0^2 - a_0a_1^2b_1 + 4a_0^2a_2b_1 + 4a_1a_2b_0b_1 - 2a_0a_2b_1^2 \\ &\quad + a_2b_1^3 - 2a_1^2b_0b_2 + 4a_0a_2b_0b_2 - 4a_2b_0b_1b_2 - a_1b_1^2b_2 + 2a_0^2b_2^2 + 4a_1b_0b_2^2, \end{aligned}}{a_2^2b_0^2 - a_1a_2b_0b_1 + a_0a_2b_1^2 + a_1^2b_0b_2 - 2a_0a_2b_0b_2 - a_0a_1b_1b_2 + a_0^2b_2^2}$$

$$\sigma_2 = \frac{\begin{aligned} &-a_0^2a_1^2 + 4a_0^3a_2 - 2a_1^3b_0 + 10a_0a_1a_2b_0 + 12a_2^2b_0^2 - 4a_0^2a_2b_1 - 7a_1a_2b_0b_1 \\ &\quad - a_1^2b_1^2 + 5a_0a_2b_1^2 - 2a_2b_1^3 + 2a_0^2a_1b_2 + 5a_1^2b_0b_2 - 4a_0a_2b_0b_2 \\ &\quad - a_0a_1b_1b_2 + 10a_2b_0b_1b_2 - 4a_1b_0b_2^2 + 2a_0b_1b_2^2 - b_1^2b_2^2 + 4b_0b_2^3. \end{aligned}}{a_2^2b_0^2 - a_1a_2b_0b_1 + a_0a_2b_1^2 + a_1^2b_0b_2 - 2a_0a_2b_0b_2 - a_0a_1b_1b_2 + a_0^2b_2^2}$$

## The Structure of $\mathcal{M}_d$ and Othe Moduli Spaces

Very little is known about the geometry of $\mathcal{M}_d$.
The space $\mathrm{Rat}_d$ is **rational**, which means that there is a generically 1-to-1 rational map

$$\mathbb{P}^{2d+1} \longrightarrow \mathrm{Rat}_d\,.$$

This is clear, since $\mathrm{Rat}_d$ is an open subset of $\mathbb{P}^{2d+1}$.
It follows that $\mathcal{M}_d$ is **unirational**, i.e., there is a generically finite-to-1 rational map

$$\mathbb{P}^{2d-2} \longrightarrow \mathcal{M}_d.$$

**Questions**. **1**. $\mathcal{M}_2 \cong \mathbb{A}_2$ is rational. Is $\mathcal{M}_3$ rational?
**2**. What do the singularities of $\mathcal{M}_d$ look like?
**3**. Let $\mathcal{M}_d(n)$ classify rational maps of degree $d$ with a marked periodic point of period $n$. For fixed $d$, is $\mathcal{M}_d(n)$ of general type for sufficiently large $n$?
**4**. Same questions for maps $\mathbb{P}^N \to \mathbb{P}^N$ for $N \geq 2$.

## Other Directions for Arithmetic Dynamics

The preceding has touched on only a few areas of current research in arithmetic dynamics. Among the topics not mentioned are:

- ***p*-adic Dynamics** There is a burgeoning field of $p$-adic dynamics that studies dynamical systems over fields such as $\mathbb{Q}_p$ and $\mathbb{C}_p$ and and iteration of maps $\phi : X \to X$ of varieties and of Berkovich spaces.
- **Iteration of Formal and *p*-adic Power Series** A fundamental problem is to classify commuting power series and describe their relationship to endomorphisms of formal groups.
- **Arithmetic Dynamics of Rational Maps** There are a number of results and conjectures regarding the arithmetic behavior of affine automorphisms $\phi : \mathbb{A}^N \xrightarrow{\sim} \mathbb{A}^N$. A classical example is the set of Hénon maps $\phi(x, y) = (y, ax + by^2 + c)$.

## Other Directions for Arithmetic Dynamics (continued)

- **Higher Dimensional Varieties**. There are many varieties other than $\mathbb{P}^N$ that admit interesting self-morphisms. An important example is the set of K3 surfaces admitting noncommuting involutions. The composition $\phi = \iota_1 \circ \iota_2$ yields interesting dynamics whose arithmetic has been studied to some extent.
- **Dynamics over Finite Fields**. Iteration of polynomial and rational functions over finite fields is an older and much studied subject. For example, the theory of permutation polynomials is well established, with 250+ reviews in MR mentioning the term (inlcuding 5 papers of Carlitz from 1953–62).
- **Twists** Maps admitting nontrivial automorphisms $\phi^f = \phi$ have nontrivial twists, analogous to twists of elliptic curves. These are maps in $K(z)$ that are $\mathrm{PGL}_2(\bar{K})$-conjugate, but not $\mathrm{PGL}_2(K)$-conjugate.

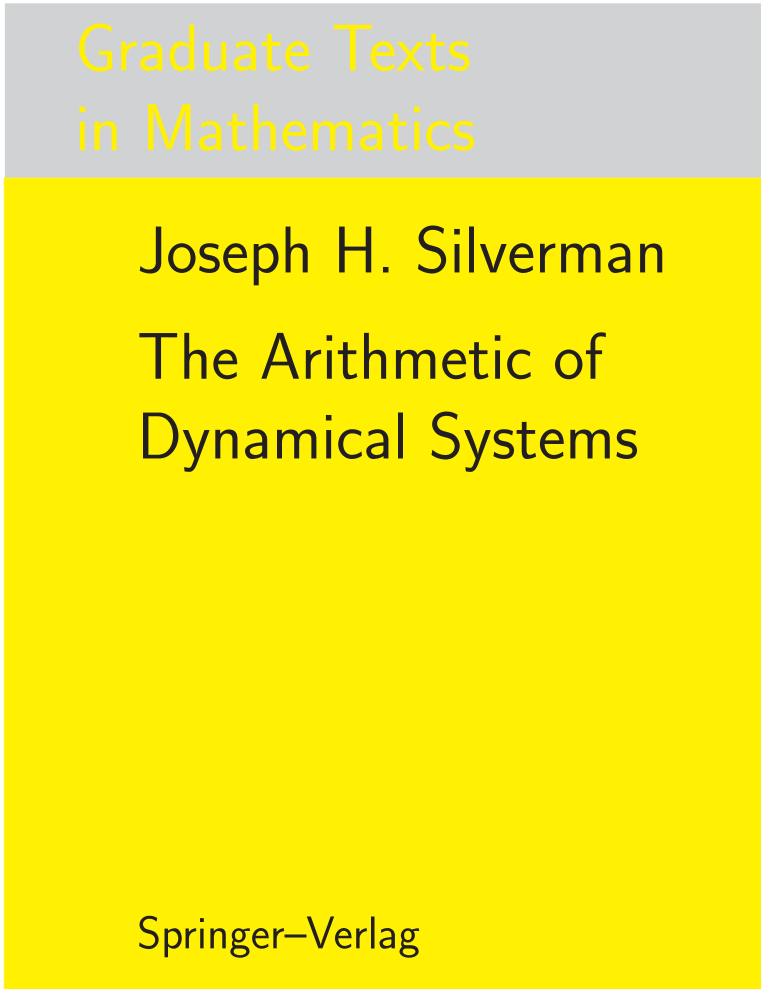
## Other Directions for Arithmetic Dynamics

- **Dynamical Equidistribution** for preperiodic points and for points of small canonical height, analogous to Raynaud's theorem (Manin–Mumford conjecture) and the Ullmo-Zhang theorem (Bogomolov conjecture) saying that subvarieties of group varieties contain few torsion points and few points of small canonical height.

- And last, but far from least, are the many beautiful interactions of number theory and dynamics in the study of **Lie Groups and Homogeneous Spaces** and in **Ergodic Theory**.

## And In Conclusion, . . . A Blantant Advertisement

For those who are interested in learning more about arithmetic dynamics, I've written an introductory graduate textbook that will be available sometime in the next couple of months.




**www.math.brown.edu/~jhs/ADSHome.html**